Testing monotonicity / lower bounds from communication complexity

List $7, 9, 2, 3, 7, 11, 19 \ldots n$

Q: Is list sorted or far from being sorted. (ε-far)

ε-far : must change values of $\geq \varepsilon n$ entries to get a monotone (non-decreasing seq.)

Q: Can test monotonicity with $o(n)$ queries?

Bad (natural) test: repeat t times

Basic test $\left\{ \begin{array}{l} \text{pick } i \text{ u.ar} \\ \text{if } \ell_i > \ell_{i+1} \text{ rej} \end{array} \right.$

accept.

Bad list for test:

$$\langle \frac{n}{2}, \frac{n}{2}+1, \ldots, \underbrace{n, 1}_{n}, \underbrace{\overset{i}{2}, 3}_{n}, \ldots, \underset{n}{n/2} \rangle$$

$$\text{dist}(\text{sortedness}) = \frac{1}{2}$$

$$\Pr[\text{basic test rej}] = \frac{1}{n-1}$$

So must make $t = \Omega(n)$ queries to get pr test rejects $\geq 2/3$

Bad test 2: repeat t times

Basic test $\left\{ \begin{array}{l} \text{pick } i < j \text{ u.a.r.} \\ \text{if } l_i > l_j \text{ rej.} \end{array} \right.$

Accept.

$\exists$ bad sequence ℓ s.t.

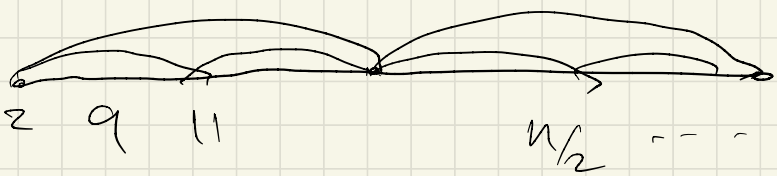
$$\text{dist}(\ell, \text{sortedness}) = \frac{1}{4}$$

$$\Pr[\text{Basic test rej}] = o(1/n)$$

(HW)

A good test: via 2-spanners.

$\Rightarrow O(\log n)$ query complexity.



pick random edges of a 2-spanner.

$\exists$ 2-spanner of size $O(n \log n)$

$\Rightarrow$ test with $O(\log n)$ queries.

(Recall HW1)

Generalizations

Testing sortedness/ monotonicity

Above:

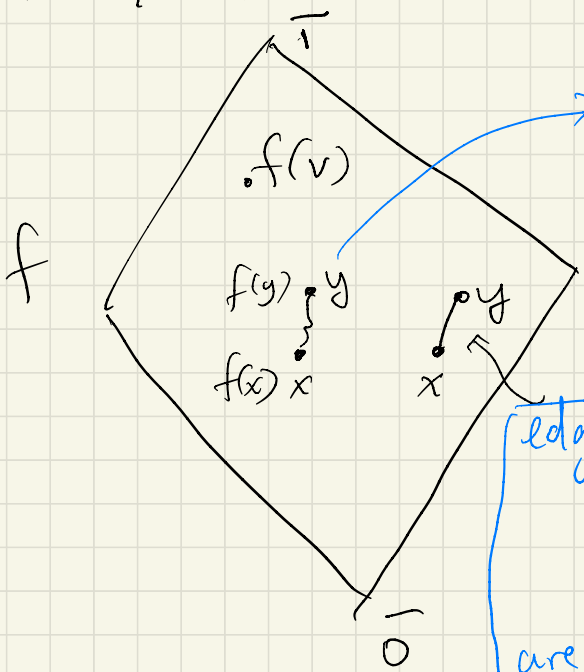
$f : [n] \to \{0,1\}$

$f : [n] \to \mathbb{R}$ discrete range/ bounded

Is f monotone or ε-far?

the hypercube

$f : \{0,1\}^n \to \{0,1\}$



$x \prec y$ iff $\forall i \in [n]$: $x_i \le y_i$

edge (x,y) if for some $i \in [n]$ $x_{-i} = y_{-i}$ (the values of x & y are same at all posns. except i)

$x_i = 0$; $y_i = 1$.

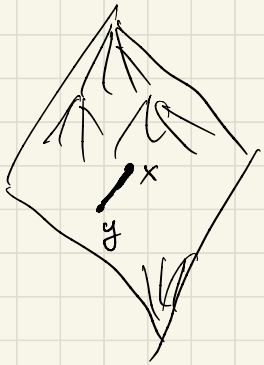
Q: Is $f(x) \le f(y) \quad \forall\, x \prec y$?
or is f ε-far from monotone (non-decreasing)

- ε-far means f must be changed in $\varepsilon \cdot 2^n$ many positions to get to a monotone func.

$$f : \{0,1\}^n \longrightarrow R$$

R is bounded / discrete

Natural test: (edge tester)



- pick a random edge $(x,y) \in H_n$
  $x \prec y$
  & check if $f(x) \le f(y)$
  acc / rej.

Thm (Goldreich et al 2000)

The edge tester is a $\Theta(n/\varepsilon)$-query tester for mononot on Boolean hypercube.

(we won't show it here)

Khot et al 2015 tight

$$\tilde{O}(\sqrt{n}/\varepsilon^2) \text{ bound.}$$

(improving Chakrabarti & Seshadri, Chen et al)

Pf idea: Path tester: pick random $x \in H_n$

pick random walk from x that ends at y.

compare $f(x)$ with $f(y)$.

Larger ranges:

$f : \{0,1\}^n \to R$ dep on n.

Goldreich et al $O(n|R|/\varepsilon)$ query

Dodis et al 2000 $O(n/\varepsilon \log |R|)$

We'll show an $\Omega(n)$ lower bound

for $R = \Omega(\sqrt{n})$.

Thm: Blais Brody Matulef 2012.

Every 2-sided, adaptive tester for monot. of $f: \{0,1\}^n \to R$, with $R = \Omega(n)$ must make $\Omega(n)$ queries to distinguish fns that are monot. from fns that are $1/8$-far from monot.

Pf: Reduction from variant of Disjointess, in particular from Unique-Disj.

Unique-Disj: Alice has A, Bob has B

$A, B \subseteq \{1, \dots n\}$.

Distinguish bet ① $A \cap B = \phi$
② $|A \cap B| = 1$

don't care abt. answer in other cases.

Razborov: $R(\text{Unique-Disj}) = \Omega(n)$

Main Lemma: Fix $A, B \subseteq \{1, \dots n\}$.

Define: $h_{A,B}(S) = 2|S| + \underbrace{(-1)^{|A \cap S|} + (-1)^{|B \cap S|}}_{\max = R.}$

$h_{AB} : [2^n] \to \mathbb{Z}$

Then ① if $A \cap B = \phi$ then h_{AB} is monotone

② if $|A \cap B| = 1$ then h_{AB} is $1/8$-far from monot.

Claim: Lemma $\Rightarrow$ Thm.

Given A, B Alice & Bob will run a comm. protocol that simulates the tester for monotocity on h_{AB}.

Alice & Bob use public randomness to select queries.

To simulate oracle access each has to compute

$$h_{AB}(S) = 2|S| + (-1)^{|A \cap S|} + (-1)^{|B \cap S|}.$$

Alice can compute $h_{AB}(S)$ using A, S & a bit communicated by Bob: $(-1)^{|B \cap S|}$

Same for Bob: needs bit $(-1)^{|A \cap S|}$ from Alice.

So each query to the oracle can be computed by both Alice & Bob with 2 bits of communication.

Alice & Bob continue simmulating the tester for monot on h_{AB} locally & & in sync., communicating 2bit/query.

At end of protocol they output $A \cap B = \phi$ if test outputs "monotone"; else $|A \cap B| = 1$ if test outpts "far".

So a tester with Q queries $\Rightarrow$ communic protocol with $2Q$ bits communicated.

By lb. on comm cost of unique disj. Test must make $\Omega(n)$ queries ✓.

Pf of Main Lemma.

$$h_{AB}(S) = 2|S| + (-1)^{|S\cap A|} + (-1)^{|S\cap B|}$$

1) $A \cap B = \phi$. WTS h_{AB} is mono not.

i.e. $\forall\ S \subseteq \{1 \ldots n\}$

$i \notin S$

$$h_{AB}(S) \leq h_{AB}(S \cup \{i\})$$

$$\underline{h_{AB}(S \cup \{i\})} - \underline{h_{AB}(S)} = 2 + \cancel{(-1)^{|S\cup\{i\}\cap A|}} + (-1)^{|S\cup\{i\}\cap B|}$$

$$- \cancel{(-1)^{|S\cap A|}} - (-1)^{|S\cap B|}$$

Note if $A \cap B = \phi \Rightarrow i \notin A$ or $i \notin B$.

Say $i \in A$, then $|S \cap A| = |(S \cup \{i\}) \cap A|$

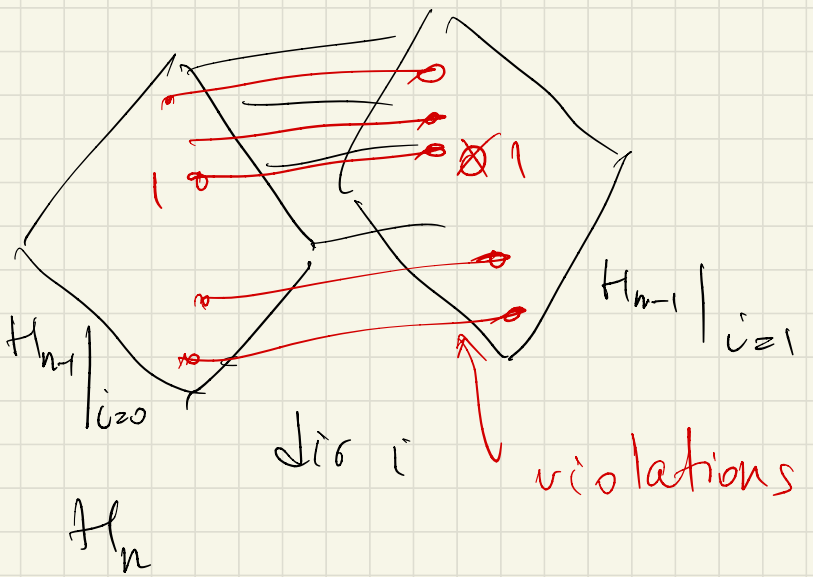
$$= 2 + (-1)^{|S\cup\{i\}\cap B|} - (-1)^{|S\cap B|} \geq 0$$

$\forall$ setting of S & B.

2) $|A \cap B| = 1$

Let $A \cap B = \{i\}$.

We'll show $\exists$ $\frac{2^n}{8}$ many disjoint edges in direction i that are violations to monotonicity.



So we'll have to fix at least one value of f per edge to get to monot, which in turn would imply f is $1/8$-far.

Recall

$$h_{AB}(S \cup \{i\}) - h_{AB}(S) = 2 + (-1)^{|(S\cup\{i\})\cap A|} + (-1)^{|(S\cup\{i\})\cap B|} - (-1)^{|S\cap A|} - (-1)^{|S\cap B|}$$

$$< 0$$

?

Let $S \subseteq \{1, \dots n\} - \{i\}$.

Assume $|S \cap A|$ & $|S \cap B|$ are even.

Then $|A \cap (S \cup \{i\})|$ is odd.

Similarly $|B \cap (S \cup \{i\})|$ is odd.

(Recall $A \cap B = \{i\}$).

$$h_{AB}(S \cup \{i\}) - h_{AB}(S) = 2 - 1 - 1 - 1 - 1 = -2 < 0$$

So, such S's give violation of monotonicity on edge $(S, S \cup \{i\})$

How many $S \subseteq \{1, \dots n\} / \{i\}$

s.t. $|A \cap S|$ & $|B \cap S|$ are even?

Claim:

$$\Pr_{S \subseteq \{1,2 \dots n\} / \{i\}} \left[|A \cap S| \text{ \& } |B \cap S| \text{ are even} \right] \geq 1/4$$

Pf. A fixed

$$\Pr_{S \subseteq \{1 \dots n\} / \{i\}} \left[|A \cap S| \text{ is even} \right] = ?$$

If $A = \{i\}$ $\quad |A \cap S| = \phi$ always.

So $\Pr[\dots] = 1$

ow. if $A \neq \{i\}$.

$$\Pr_S [\quad] = 1/2$$

in either case $\geq 1/2$

Same for B.

$$\forall B \quad \Pr_{S \subseteq \{1..n\}/\{i\}} \left[|B \cap S| \text{ is even} \right] \geq \frac{1}{2}$$

Because $A \cap B = \{i\}$.

events $|A \cap S|$ even, $|B \cap S|$ even } are independent

So claim follows.

Finally, how many $S \subseteq \{1..n\}/\{i\}$ are s.t. both $|A \cap S|$ & $|B \cap S|$ even?

$$\frac{1}{4} \cdot 2^{n-1} = \frac{1}{8} \cdot 2^n \text{ edges are}$$

violated $\Rightarrow$ $\text{dist}(h_{AB}, \text{monotonicity}) \geq \frac{1}{8}$ ✓