Testing monotonicity / lower bounds from communication complexity

List 7, 9, 2, 3, 7, 11, 19, \ldots

Q: Is list sorted or far from being sorted. (\varepsilon-far)

\varepsilon-far: must change values of \geq \varepsilon n entries to get a monotone (non-decreasing seq.)

Q: Can test monotonicity with o(n) queries?

Bad (natural) test: repeat t times

Basic test:
\begin{align*}
\text{pick } i \text{ u.a.r.} \\
\text{if } l_i > l_{i+1} \text{ rej} \\
\text{accept.}
\end{align*}
Bad list for test:

\[ \left< \frac{n}{2}, \frac{n}{2} + 1, \ldots, \frac{n}{2}, \frac{n}{2} + 1, \ldots, \frac{n}{2} \right> \]

\[ \text{dist (sort edness)} = \frac{1}{2} \]

Do \[ \left| \text{basic test rej} \right| = \frac{1}{n-1} \]

So must make \( t = \Omega(n) \) queries to get pr test rejects > \( \frac{2}{3} \)

Bad test 2: repeat \( t \) times

Basic test \[
\begin{array}{l}
\{ \text{pick } i < j \text{ u.a.r.} \\
\text{if } l_i > l_j \text{ rej.} \\
\text{Accept.} \\
\} 
\end{array}
\]
I bad sequence $x$ s.t.

$$
\text{dist}(x, \text{sortedness}) = \frac{1}{4}
$$

$$
\Pr[\text{Basic test rej}] = o\left(\frac{1}{n}\right)
$$

---

A good test: via 2-spanners.

$$
\Rightarrow O(\log n) \text{ query complexity.}
$$

Pick random edges of a 2-spanner.

$$
\exists 2\text{-spanner of size } O(n \log n)
$$

$$
\Rightarrow \text{test with } O(\log n) \text{ queries.}
$$

(Recall HW!)
Generalizations

Testing sortedness/monotonicity

Above:
- $f : [n] \rightarrow \{0, 1\}$
- $f : [n] \rightarrow \{0, 1, 2\}$

Is $f$ monotone or range/bounded?

The hypercube

$f : \{0, 1\}^n \rightarrow \{0, 1\}^n$

$f(x) \geq y$ if

$\forall i \in [n]. \ x_i \leq y_i$

Edge $(x, y)$ if for some $i \in [n]$

$\quad x_i = y_i$

$\quad (\text{the values of } x \text{ and } y \text{ are same at all posns. except } i)$

$x_i = 0 \ ; \ y_i = 1$
Q: Is \( f(x) \leq f(y) \) \( \forall x \leq y \)?

or is \( f \) \( \varepsilon \)-far from monotone (non-decreasing)?

- \( \varepsilon \)-far means \( f \) must be changed in \( \varepsilon \cdot 2^n \) many positions to get to a monotone func.

\( f: \Sigma^* \rightarrow R \)

\( R \) is bounded / discrete

Natural test: (edge tester)

- pick a random edge \((x, y) \in \Sigma^*_n\)

\( x < y \)

\( f \) check if \( f(x) \leq f(y) \)

\( \text{acc} \) (\( \text{rej} \)
The edge tester is a $\Theta(1/\varepsilon)$-query tester for monotone on Boolean hypercube.

(we won't show it here)

Khot et al. 2015 tight $\widetilde{O}(\sqrt{n}/\varepsilon^2)$ bound.

(improving Chakraborti & Seshadri, Chen et al.)

Pf idea: Path tester:
- pick random $x \in H_n$
- pick random walk from $x$ that ends at $y$.
- compare $f(x)$ with $f(y)$.
Larger ranges:

\[ f: \mathbb{R}^n \rightarrow \mathbb{R} \]

dep on \( n \).

Goldreich et al. 0(\( n \log R / \varepsilon \)) query

Dodis et al. 2003 0(\( n \log \log R / \varepsilon \))

We'll show an \( \Omega(n) \) lower bound

for \( R = \Omega(\sqrt{n}) \).

Every 2-sided, adaptive tester for monot. of $f: \Sigma^* \rightarrow R$, with $R = \Omega(n)$ must make $\Omega(n)$ queries to distinguish $f$s that are monot. from $f$s that are $1/8$-far from monot.

**Pf:** Reduction from variant of Disjointness, in particular from Unique-Disj.

**Unique-Disj:** Alice has $A$, Bob has $B$

$A \cap B \subseteq \{1, \ldots, n\}$

Distinguish bet

1. $A \cap B = \emptyset$
2. $|A \cap B| = 1$

? don't care abt. answer in other cases.
Razborov: \( R(\text{Unique Disj}) = \Omega(n) \)

**Main Lemma:** Fix \( A, B \subseteq \{1, \ldots, n\} \).

Define: \( h_A \mid B \) \( (S) = 2|S| + (-1)^{|\text{Ans}}| + (-1)^{|\text{Bns}}| \)

\( h_{A \mid B} : [2^n] \rightarrow \mathbb{Z} \) \hspace{1cm} \text{max} = R.

Then

1. If \( A \cap B = \emptyset \) then \( h_{A \mid B} \) is monotone.

2. If \( |A \cap B| = 1 \) then \( h_{A \mid B} \) is \( \frac{1}{8} \) far from monotone.

**Claim**: Lemma \( \Rightarrow \) Thm.

Given \( A, B \), Alice & Bob will run a comm. protocol that simulates the tester for monotonicity on \( h_{A \mid B} \).
Alice & Bob use public randomness to select queries. To simulate oracle access each has to compute

\[ h_{\text{Ab}}(s) = 2 \| s \| + (-1)^{A_{\text{Ab}}} + (-1)^{B_{\text{Ab}}} \]

Alice can compute \( h_{\text{AAb}}(s) \) using \( A, s \) & a bit communicated by Bob: \((-1)^{B_{\text{Ab}}} \)

Same for Bob: needs bit \((-1)^{A_{\text{Ab}}} \) from Alice.

So each query to the oracle can be computed by Alice & Bob with 2 bits of communication.
Alice & Bob continue simulating the tester for monotone $h_{AB}$ locally & in sync., communicating $2$ bits/query. At end of protocol they output $A \land B = \emptyset$ if test outputs "monotone," else $|A \land B| = 1$ if test outputs "far.

So a tester with $Q$ queries $\Rightarrow$ communication protocol with $2Q$ bits communicated.

By lb. on comm cost of unique disj. Test must make $O(n)$ queries.
Pf of Main Lemma. \[ h_{\text{AB}}(S) = 2|S| + (-1)^{|S|} + (-1) \]

1) \( A \cap B = \emptyset \). WTS \( h_{\text{AB}} \) is monot.

i.e. \( \forall S \subseteq \{1, \ldots, n\} \)
\[ i \notin S \]
\[ h_{\text{AB}}(S) \leq h_{\text{AB}}(S \cup \{i\}) \]

\[ h_{\text{AB}}(S \cup \{i\}) - h_{\text{AB}}(S) = 2 + (-1)^{|S|} + (-1) \]

Note if \( A \cap B = \emptyset \Rightarrow i \notin A \text{ or } i \notin B. \]
Say \( i \in A \), then \( |S \cap A| = |(S \cup \{i\}) \cap A| \)
\[ = 2 + (-1)^{|S|} + (-1) \]
\[ > 0 \]
\[ \forall \text{ setting of } S \cap B. \]
Let \( A \cap B \neq \emptyset \) and \( A \subset H \cap I \).

So we'll have to fix at least one value of \( f \) for each edge. In direction 1, there are violations to monotonicity.

We'll show \( |A \cap B| = 2 \) disjoint monomials.

Let \( |A \cap B| = 2 \).

If we know \( |A \cap B| = 2 \), then we have violations to monotonicity.

If we fix one value of \( f \) per edge, then each monomial is fixed at least one value of \( f \).
Recall

\[ h_{AB}(S \cup \xi) - h_{AB}(S) = 2 + (-1) + (-1) + (-1) \leq 0 \]

Let \( S \leq \xi \).

Assume \(|S \cup A|\) and \(|S \cup B|\) are even.

Then \( |A \cap (S \cup \xi)| \) is odd.

Similarly \( |B \cap (S \cup \xi)| \) is odd.

(Recall \( A \cap B = \xi \).)

\[ h_{AB}(S \cup \xi) - h_{AB}(S) = 2 - 1 - 1 - 1 = -2 \leq 0 \]

So, such \( S \)'s give violation of monotonicity on edge \((S, S \cup \xi)\).
How many $S \subseteq \Xi_1 \cup \Xi_2 \cup \Xi_3$ 

$s.t. |A \cap S| \& (|B \cap S|)$ are even?

Claim:

$$\Pr \left[ (|A \cap S|) \& (|B \cap S|) \text{ are even} \right] \geq \frac{1}{4}$$

$S \subseteq \Xi_1 \cup \Xi_2 \cup \Xi_3.$

Pf. A fixed

$$\Pr \left[ |A \cap S| \text{ is even} \right] = ?$$

$S \subseteq \Xi_1 \cup \Xi_2 \cup \Xi_3.$

If $A = \Xi_3$ and $|A \cap S| = \emptyset$ always

so $\Pr \left[ |A \cap S| \right] = 1$

ow, if $A \neq \Xi_3$.

$$\Pr \left[ |A \cap S| \right] \geq \frac{1}{2}$$

$$s.t. |A \cap S| \& (|B \cap S|)$$
Same for B.

\[
\forall \beta \quad \Pr \left[ |B_N S| \text{ is even} \right] \geq \frac{1}{2}
\]

\[S \leq \frac{3}{\sqrt{n}}/\xi_i \zeta \]

Because \( A \cap B = \xi_i \zeta \).

Events \(|A N S| \text{ even}\) are independent \(|B N S| \text{ even}\)

So claim follows.

Finally, how many \( S \leq \frac{3}{\sqrt{n}}/\xi_i \zeta \) are set both \(|A N S| \text{ and } |B N S| \text{ even}\)?

\[
\frac{1}{4} \cdot 2^{n-1} = \frac{1}{8} \cdot 2^n \quad \text{edges are violated} \Rightarrow \text{dist} (h_{A \cap B}, \text{monotonicity}) \geq \frac{1}{8}
\]