

Identification and Model Testing in Linear Structural Equation Models using Auxiliary Variables

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Abstract

We developed a novel approach to identification and model testing in linear structural equation models (SEMs) based on auxiliary variables (AVs), which generalizes a widely-used family of methods known as instrumental variables. The identification problem is concerned with the conditions under which causal parameters can be uniquely estimated from an observational, non-causal covariance matrix. In this paper, we provide an algorithm for the identification of causal parameters in linear structural models that subsumes previous state-of-the-art methods. In other words, our algorithm identifies strictly more coefficients and models than methods previously known in the literature. Our algorithm builds on a graph-theoretic characterization of conditional independence relations between auxiliary and model variables, which is developed in this paper. Further, we leverage this new characterization for allowing identification when limited experimental data or new substantive knowledge about the domain is available. Lastly, we develop a new procedure for model testing using AVs.

1 Introduction

The problem of estimating causal effects is one of the fundamental problems in the data-driven sciences. In order to estimate a causal effect, the desired effect must be *identified* or uniquely expressible in terms of the probability distribution over the available data. Causal effects are identified by design in randomized control trials, but in many applications, such experiments are not possible. When only observational data is available, determining whether a causal effect is identified requires modeling the underlying causal structure, which is generally done using *structural equation models* (SEMs) (also called *structural causal models*) (Pearl, 2009; Bareinboim and Pearl, 2016).

A structural equation model consists of a set of equations that describe the underlying data-generating process for a set of variables. While SEMs, in their most general, non-parametric form do not require any assumptions about the form of these functions, in many fields, including machine learning, psychology, and the social sciences, linear SEMs

are used. A linear SEM consists of a set of equations of the form, $X = \Lambda X + U$, where $X = [x_1, \dots, x_n]^t$ is a vector containing the model variables, Λ is a matrix containing the *coefficients* of the model, and Λ_{ij} represents the direct effect of x_i on x_j , and $U = [u_1, \dots, u_n]^t$ is a vector of normally distributed error terms, which represents omitted or latent variables.¹ The matrix Λ contains zeroes on the diagonal, and $\Lambda_{ij} = 0$ whenever x_i is not a cause of x_j . The covariance matrix of X will be denoted by Σ and the covariance matrix over the error terms, U , by Ω . In this paper, we will restrict our attention to *semi-Markovian* models (Pearl, 2009), models where the rows of Λ can be arranged so that it is lower triangular, and the corresponding graph is acyclic.

When modeling using SEMs, researchers typically specify the model by setting certain entries of Λ and Ω to zero (i.e. exclusion and independence restrictions), while leaving the rest of the entries as free parameters to be estimated from data². Restricting a particular entry Λ_{ij} to zero reflects the assumption that Y_i has no direct effect on Y_j . Similarly, restricting Ω_{ij} to zero reflects the assumption that there are no unobserved common causes of both Y_i and Y_j . Once the parameters are estimated, causal effects (as well as counterfactual quantities) can be computed from the structural coefficients directly (Pearl, 2009; Chen and Pearl, 2014). However, in order to be estimable from data, a parameter must first be identified. In some cases, the modeling assumptions are not strong enough, and there are multiple, often infinite, values for the parameter that are consistent with the observed data. As a result, two fundamental problems in SEMs are to identify and estimate the model parameters and to test the underlying assumptions that enable identification.

The problem of identification has been studied extensively by econometricians and social scientists (Fisher, 1966; Bowden and Turkington, 1984; Bekker et al., 1994; Rigdon, 1995)

¹Instrumental and auxiliary variables can also be used when normality is not assumed, but to simplify the proofs in the paper, we will, as is commonly done by empirical researchers, assume normality.

²There are a number of algorithms for discovering the model structure from data (Spirites et al., 2000; Shimizu et al., 2006; Pearl, 2009; Zhang and Hyvärinen, 2009; Mooij et al., 2016). However, it is only in very rare instances that these methods are able to uniquely determine the model structure. As a result, model specification generally utilizes knowledge about the domain under study.

and more recently by the AI and statistics communities using graphical methods (Spirtes et al., 1998; Tian, 2007, 2009; Brito and Pearl, 2002a,c, 2006; Bareinboim and Pearl, 2016). To our knowledge, the most general, efficient algorithm for model identification is the g-HT algorithm given by Chen (2016) combined with ancestor decomposition (Drton and Weihs, 2016). This method generalizes the half-trek algorithm of Foygel et al. (2012) and utilizes ancestor decomposition, which expands on an idea by Tian (2005) where the model is decomposed into simpler sub-models. Graphical methods have also been applied to the problem of testing the causal assumptions embedded in an SEM. For example, d-separation (Pearl, 2009) and overidentification (Pearl, 2004; Chen et al., 2014) provide the means to discover testable implications of the model, which can be used to test it against data.

Despite decades of attention and work from diverse fields, the identification problem³ has still not been efficiently solved⁴. There are identifiable parameters and models that none of the above methods are able to identify. Similarly, there are testable implications of SEMs that the above methods are unable to detect. One promising avenue to aid in both tasks are *auxiliary variables* (Chen et al., 2016). Each of the aforementioned methods for identification and model testing only utilizes restrictions on the entries of Λ and Ω to zero. Auxiliary variables can be used to incorporate knowledge of non-zero coefficient values into existing methods for identification and model testing. These coefficient values could be obtained, for example, from a previously conducted randomized experiment, from substantive understanding of the domain, or even from another identification technique. The intuition behind auxiliary variables is simple: if the coefficient from variable w to z , β , is known, then we would like to remove the direct effect of w on z by subtracting it from z . This removal eliminates confounding paths through w and is performed by creating a variable $z^* = z - \beta w$, which is used as a proxy for z . In many cases, z^* allows the identification of parameters or testable implications using existing methods when z could not.

Chen et al. (2016) demonstrated how auxiliary variables could be utilized in simple instrumental sets (instrumental sets that do not utilize conditioning to block spurious paths) (Bruto and Pearl, 2002a; van der Zander et al., 2015) and proved that any model identifiable using the g-HT algorithm is also identifiable using auxiliary simple instrumental sets.

Since auxiliary variables allow knowledge of non-zero coefficient values to be incorporated into existing methods for identification, they are also directly applicable to the problem of z-identification (Bareinboim and Pearl, 2012), in which partial experimental data is available. Additionally, the cancellation of paths that results from adding an AV may result in conditional independence constraints between the AV and

³To be precise, we are referring to the problem of identification almost everywhere (Bruto and Pearl, 2002b), also called generic identification (Foygel et al., 2012).

⁴An exhaustive procedure can be obtained using Gröbner bases methods (Foygel et al., 2012). However, these methods are computationally intractable for anything but the smallest of graphs.

other variables that can be used to test the model.

In this paper, we generalize the results of Chen et al. (2016) and demonstrate how auxiliary variables can be utilized in generalized instrumental sets, which allow for conditioning to block spurious paths. We prove that, unlike auxiliary simple instrumental sets, this generalization *strictly* subsumes the g-HT algorithm. Additionally, we introduce quasi-instrumental sets, which utilize auxiliary variables to identify coefficients when partial experimental data is available. Quasi-instrumental sets are incorporated into our identification algorithm, allowing it to better address the problem of z-identification. To our knowledge, this algorithm is the first systematic method for tackling z-identification in linear systems. We also demonstrate how auxiliary instrumental sets and quasi-instrumental sets can be used to derive over-identifying constraints, which can be used to test the model specification against data. Moreover, we prove that these overidentifying constraints subsume conditional independence constraints among auxiliary variables. Lastly, we discuss related work, showing how auxiliary IVs are able to unite a variety of disparate methods under a single framework.

2 Preliminaries

The causal graph or path diagram of an SEM is a graph, $G = (V, D, B)$, where V are nodes or vertices, D directed edges, and B bidirected edges. The nodes represent model variables. Directed edges encode the direction of causality, and for each coefficient $\Lambda_{ij} \neq 0$, an edge is drawn from x_i to x_j . Each directed edge, therefore, is associated with a coefficient in the SEM, which we will often refer to as its structural coefficient. Additionally, when it is clear from context, we may abuse notation slightly and use coefficients and directed edges interchangeably. The error terms, u_i , are not shown explicitly in the graph. However, a bidirected edge between two nodes indicates that their corresponding error terms may be statistically dependent while the lack of a bidirected edge indicates that the error terms are independent.

We will use standard graph terminology with $Pa(y)$ denoting the parents of y , $Anc(y)$ denoting the ancestors of Y , $De(y)$ denoting the descendants of y , and $Sib(y)$ denoting the siblings of y , the variables that are connected to y via a bidirected edge. $He(E)$ denotes the heads of a set of directed edges, E , while $Ta(E)$ denotes the tails. Additionally, for a node v , the set of edges for which $He(E) = v$ is denoted $Inc(v)$. Lastly, we will utilize d-separation (Pearl, 2009).

We will use $\sigma(x, y|W)$ to denote the partial covariance between two random variables, x and y , given a set of variables, W , and $\sigma(x, y|W)_G$ as the partial covariance between random variables x and y given W implied by the graph G . We will assume without loss of generality that the model variables have been standardized to mean 0 and variance 1.

Definition 1. For a given unblocked (given the empty set) path, π , from x to y , $Left(\pi)$ is the set of nodes, if any, that has a directed edge leaving it in the direction of x in addition to x . $Right(\pi)$ is the set of nodes, if any, that has a directed edge leaving it in the direction of y in addition to y .

For example, consider the path $\pi = x \leftarrow v_1^L \leftarrow \dots \leftarrow v_k^L \leftarrow v^T \rightarrow v_j^R \rightarrow \dots \rightarrow v_1^R \rightarrow y$. In this case, $\text{Left}(\pi) = \cup_{i=1}^k v_i^L \cup \{x, v^T\}$ and $\text{Right}(\pi) = \cup_{i=1}^j v_i^R \cup \{y, v^T\}$. v^T is a member of both $\text{Right}(\pi)$ and $\text{Left}(\pi)$.

Definition 2. A set of paths, π_1, \dots, π_n , has no sided intersection if for all $\pi_i, \pi_j \in \{\pi_1, \dots, \pi_n\}$ such that $\pi_i \neq \pi_j$, $\text{Left}(\pi_i) \cap \text{Left}(\pi_j) = \text{Right}(\pi_i) \cap \text{Right}(\pi_j) = \emptyset$.

Wright’s rules (Wright, 1921) allow us to equate the model-implied covariance, $\sigma(x, y)_M$, between any pair of variables, x and y , to the sum of products of parameters along unblocked paths between x and y .⁵ Let $\Pi = \{\pi_1, \pi_2, \dots, \pi_k\}$ denote the unblocked paths between x and y , and let p_i be the product of structural coefficients along path π_i . Then the covariance between variables x and y is $\sum_i p_i$.

Lastly, we define auxiliary variables and the augmented graph.

Definition 3 (Auxiliary Variable). Given a linear SEM with graph G and a set of edges E whose coefficient values are known, an auxiliary variable is a variable, $z^* = z - \sum_i e_i t_i$, where $\{e_1, \dots, e_k\} \subseteq E \cap \text{Inc}(z)$ and $t_i = \text{Ta}(e_i)$ for all $i \in \{1, \dots, k\}$.

If not otherwise specified, z^* refers to the auxiliary variable, $z - c_1 t_1 - \dots - c_l t_l$, where $\{c_1, \dots, c_l\}$ are the coefficients of $E \cap \text{Inc}(z)$ and E is the set of directed edges whose coefficient values are known. In other words, z^* is the auxiliary variable for z where as many known coefficients are subtracted out as possible. Chen et al. (2016) demonstrated that the covariance between any auxiliary variables and model variables can be computed using Wright’s rules on the augmented graph, defined below.

Definition 4. (Chen et al., 2016) Let M be a linear SEM with graph G and a set of directed edges E such that their coefficient values are known. The E -augmented model, M^{E+} , includes all variables and structural equations of M in addition to new auxiliary variables, y_1^*, \dots, y_k^* , one for each variable in $\text{He}(E) = \{y_1, \dots, y_k\}$ such that the structural equation for y_i^* is $y_i^* = y_i - \Lambda_{X_i y_i} T_i^t$, where $X_i = \text{Ta}(E) \cap \text{Pa}(y_i)$, for all $i \in \{1, \dots, k\}$. The corresponding augmented graph is denoted G^{E+} .

For example, consider Figure 1a. If the value of β is known, we can generate an auxiliary variable $x^* = x - \beta t$. The β -augmented graph $G^{\beta+}$ is depicted in Figure 1b. In some cases, x^* allows the identification of coefficients and testable implications using existing methods when x could not, due to the fact that the back-door paths from x to y that go through β cancel with the back-door paths from x^* to y that go through $-\beta$. This can be seen by expressing the covariance of x^* and y in terms of the model parameters using Wright’s rules.

⁵Wright’s rules characterize the relationship between the covariance matrix and model parameters. Therefore, any question about identification using the covariance matrix can be decided by studying the solutions for this system of equations. However, since these equations are polynomials and not linear, it can be very difficult to analyze identification of models using Wright’s rules.

3 Auxiliary and Quasi-Instrumental Sets

Two, perhaps the most common, methods for estimating causal effects are OLS regression and two-stage least-squares (2SLS) regression. Both of these methods assume that the underlying causal relationships between variables are linear, in addition to other causal assumptions that guarantee identification. The *single-door criterion* (Pearl, 2009) graphically characterizes when the assumptions sufficient to estimate a causal effect using regression are satisfied in a linear SEM. Similarly, Brito and Pearl (2002a) gave a graphical characterization for when a variable z qualifies as an IV so that 2SLS regression provides a consistent estimate of the causal effect. In this section, we give a graphical criterion for when AVs can be utilized in generalized instrumental sets, which extends both the single-door criterion and IVs. Additionally, we introduce quasi-instrumental sets, which utilize AVs to better address the problem of z -identification.

First, we give a simple graphical criterion for when an AV would be conditionally independent of another variable, which will allow us to incorporate AVs into instrumental sets, as well as other identification and model testing methods that require the ability to detect conditional independence in the graph.

Theorem 1. Given a linear SEM with graph G , where $E \subseteq \text{Inc}(z)$ is a set of edges whose coefficient values are known, if $W \cup \{y\}$ does not contain descendants of z and G_{E-} represents the graph G with the edges for E removed, then $(z^* \perp\!\!\!\perp y | W)_{G^{E+}}$ if and only if $(z \perp\!\!\!\perp y | W)_{G_{E-}}$.⁶

Proof. Proofs for all theorems and lemmas can be found in the Appendix. \square

Next, we demonstrate how AVs can be incorporated into generalized instrumental sets, defined below.

Theorem 2. (Bruto and Pearl, 2002a) Given a linear model with graph G , the coefficients for a set of edges $E = \{(x_1, y), \dots, (x_k, y)\}$ are identified if there exists triplets $(z_1, W_1, p_1), \dots, (z_k, W_k, p_k)$ such that for $i = 1, \dots, k$,

- (i) $(z_i \perp\!\!\!\perp y | W_i)_{G_{E-}}$, where W does not contain any descendants of y and G_{E-} is the graph obtained by deleting the edges, E from G ,
- (ii) p_i is a path between z_i and x_i that is not blocked by W_i , and

⁶The theorem disallows descendants of the generating variable in the conditioning set. At first glance, this may appear to limit the ability to block biasing paths among AVs. However, we conjecture that if z cannot be separated from y in G , then z^* will almost surely not be independent of y given W , if W contains descendants of z . To illustrate, consider the example shown in Figure 1c. $x^* = x - \beta t$ is independent of y , as can be verified using Wright’s rules, but x^* is not independent of y given d ! An intuitive explanation for this surprising result is that conditioning on d , a descendant of x , in Figure 1c induces correlation between the error term of x and t , since x acts as a “virtual collider”. As a result, we have a “virtual path” from x^* to y , $x^* \leftarrow x \leftarrow u_x \leftrightarrow t \rightarrow y$. See Pearl (2009, p. 339) for a detailed discussion of virtual colliders.

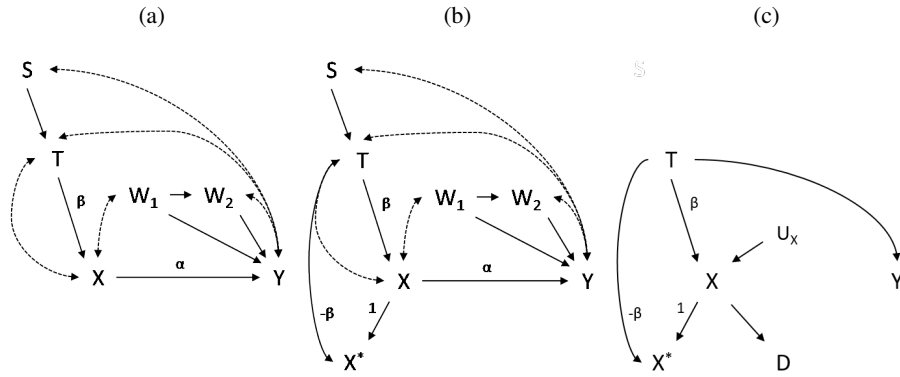


Figure 1: (a) α is not identified using IVs (b) α is identified using x^* as an auxiliary IV given w_1 (c) conditioning on descendants of x induces correlation between x^* and y

(iii) the set of paths, $\{p_1, \dots, p_k\}$ has no sided intersection.⁷

If the above conditions are satisfied, we say that Z is a generalized instrumental set for E or simply an instrumental set for E .⁸

In some cases, a variable z may not satisfy condition (i) above but an auxiliary variable z^* does. For example, in Figure 1a, we cannot identify α using Theorem 8. Blocking the path $x \leftarrow t \leftrightarrow y$ by conditioning on t opens the path, $x \leftrightarrow t \leftrightarrow y$. Moreover, we cannot use t or s in an instrumental set due to the edges $t \leftrightarrow y$ and $s \leftrightarrow y$. However, s is an IV for β , allowing us to generate an AV, $x^* = x - \beta \cdot t_1$, as in Figure 1b. Now, α can be identified using x^* as an auxiliary instrument given w_1 .

Theorem 1 tells us when (i) of Theorem 8 can be satisfied using an AV, z_i^* . We simply check whether z_i can be separated from y in $G_{E \cup E_z^-}$, where $E_z \subseteq \text{Inc}(z_i)$ is the set of z_i 's edges whose coefficient values are known. When an instrumental set includes AVs, we call the set an *auxiliary instrumental set* or *auxiliary IV set* for short.

Figure 1a also demonstrates the importance of extending the simple auxiliary instrumental sets introduced by Chen et al. (2016) to allow for conditioning. α can only be identified if we block the paths $x \leftrightarrow w_1 \rightarrow y$ and $x \leftrightarrow w_1 \rightarrow w_2 \rightarrow y$ by conditioning on w_1 .

When knowledge of coefficient values are known a priori, it may be helpful to generate an AV from the outcome variable y . For example, in Figure 2a, α cannot be identified. However, suppose that it is possible to run a surrogate experiment and randomize z . This experiment would allow us to estimate γ and generate the AV, $Y^* = Y - \gamma Z$. Now, z is not technically an instrument for α , but it can be shown that $\alpha = \frac{\tau_{Y^*Z} \cdot \gamma}{\tau_{XZ}}$. Chen et al. (2016) called such variables *quasi-instrumental variables* or *quasi-IVs* for short.

Interestingly, while quasi-IVs are valuable for the problem of z -identification, they do no better than instrumental sets

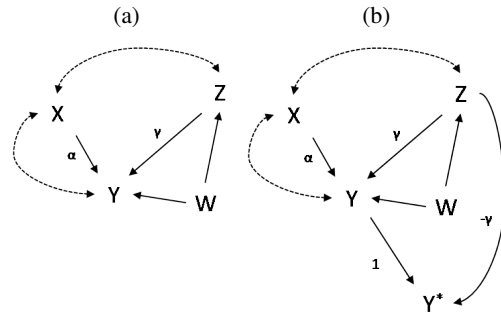


Figure 2: (a) α is not identified using IVs (b) α is identified using Z as a quasi-IV after adding auxiliary variable Y^*

when applied to the standard identification problem, where no external knowledge of coefficient values is available. For example, consider again Figure 2a. In order to use z as a quasi-IV for α , we would first have to identify γ using an IV. If such a variable existed, say z' , then we could have simply identified $\{\alpha, \gamma\}$ using the IV set $\{z, z'\}$.

Next, we formally define *quasi-instrumental sets* or *quasi-IV sets* for short. Note that auxiliary IV sets are also quasi-IV sets.

Definition 5. Given a linear SEM with graph G , a set of edges E_K whose coefficient values are known, and a set of structural coefficients $\alpha = \{\alpha_1, \alpha_2, \dots, \alpha_k\}$, the set $Z = \{z_1, \dots, z_k\}$ is a *quasi-instrumental set* if there exist triples $(z_1, W_1, p_1), \dots, (z_k, W_j, p_k)$ such that:

- (i) For $i = 1, \dots, k$, either:
 - (a) the elements of W_i are non-descendants of y , and $(z_i \perp\!\!\!\perp y | W_i)_{G_{E \cup E_y}}$ where $E_y = E_K \cap \text{Inc}(y)$.
 - (b) the elements of W_i are non-descendants of z_i and y , and $(z_i \perp\!\!\!\perp y | W_i)_{G_{E \cup E_{z_i}}}$ where $E_{z_i} = E_K \cap (\text{Inc}(z_i) \cup \text{Inc}(y))$.
- (ii) for $i = 1, \dots, k$, p_i is a path between z_i and x_i that is not blocked by W_i , where $x_i = \text{He}(\alpha_i)$, and
- (iii) the set of paths $\{p_1, \dots, p_k\}$ has no sided intersection

⁷ Brito and Pearl (2002a) provided an alternative statement of condition (iii). A proof that the two statements are, in fact, equivalent is given in the Appendix.

⁸ Note that when $k = 1$, z_1 is an IV for (x_1, y) . Further, if $z_1 = x_1$, then x_1 satisfies the single-door criterion for (x_1, y) .

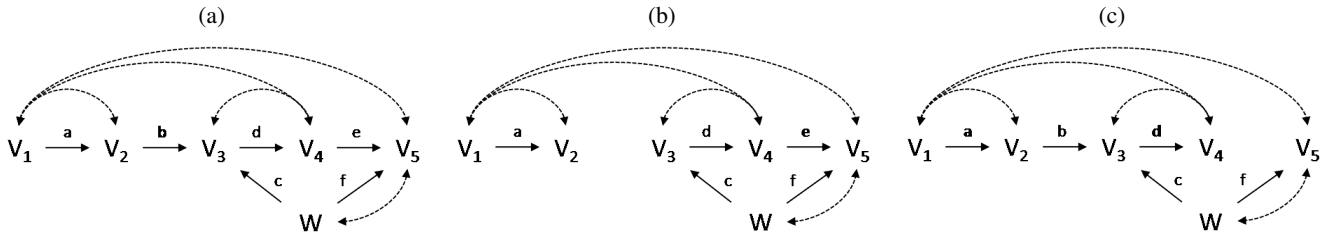


Figure 3: (a) b is identified using either v_2 or v_1 as an instrument and c is identified using w as an instrument (b) e is identified using v_3^* as an auxiliary instrument given (c) a and d are identified using v_5^* as an auxiliary instrument

Theorem 3. *If Z^* is a quasi-instrumental set for E , then E is identifiable.*

Lastly, the following corollary provides a simple graphical condition for when a single variable or AV qualifies as a quasi-IV.

Corollary 1. *Given a linear SEM with graph G , z^* is a quasi-IV for α given W if W does not contain any descendants of z , and z is an IV for α given W in $G_{E_z \cup E_y^-}$, where $E_z \subseteq \text{Inc}(z)$ and $E_y \subseteq \text{Inc}(y)$ are sets of edges whose coefficient values are known.*

Auxiliary and quasi-IV sets enable a bootstrapping procedure whereby complex models can be identified by iteratively identifying coefficients and using them to generate new auxiliary variables. For example, consider Figure 3a. First, we are able to identify b and c using IVs, but no other coefficients. Once b is identified, Corollary 1 tells us that e is identified using v_3^* since v_3 is an IV for e when the edge for b is removed (see Figure 3b). Now, the identification of e allows us to identify a and d using v_5^* , since v_5 is an IV for a and d when the edge for e is removed (see Figure 3c). This general strategy is the basis for our identification, z -identification, and model testing algorithm, described next.

4 Identification and z -Identification Algorithm

In this section, we construct an identification algorithm that operationalizes the bootstrapping approach described in Section 3. First, we describe how to algorithmically find a quasi-instrumental set for a set of coefficients E , given a set of known coefficients, IDEdges .

The problem of finding generalized instrumental sets was addressed by van der Zander and Liskiewicz (2016). They provided an algorithm, TestGeneralIVs , that determines whether a given set Z is a generalized instrumental set for a set of edges, E , that runs in polynomial time if we bound the size of the coefficient set to be identified. More specifically, their algorithm has a running time of $O((k!)^2 n^k)$, where n is the number of variables in the graph and $k = |E|$.⁹

Our method, TestQIS , given in the Appendix, generalizes TestGeneralIVs , for quasi-IV sets. FindQIS , also given in

⁹van der Zander and Liskiewicz (2016) also give an algorithm that tests whether Z is a *simple conditional instrumental sets* in $O(nm)$ time. A simple conditional instrumental set is a generalized instrumental set where $W_1 = W_2 = \dots = W_k$

the Appendix, searches for a quasi-IV set by checking all subsets of $Z \subseteq (\text{Anc}(z_i) \cup \text{Anc}(y))$ using TestQIS . It returns a quasi-IV set, as well as its conditioning sets, if one exists.

In some cases an instrumental set may not exist for C , but one exists for C' , where $C \subset C'$. Conversely, there may not be an instrumental set for C' , but there is one for $C \subset C'$. As a result, we may have to check all possible subsets of a variable's coefficients in order to determine whether a given subset is identifiable using auxiliary instrumental sets. This search can be simplified somewhat by noting that if E is a *connected edge set* (defined below) with no instrumental set, then there is no superset E' with an instrumental set.

Definition 6. (Chen et al., 2014) *For an arbitrary variable, V , let Pa_1, Pa_2, \dots, Pa_k be the unique partition of $Pa(V)$ such that any two parents are placed in the same subset, Pa_i , whenever they are connected by an unblocked path. A connected edge set with head V is a set of directed edges from Pa_i to V for some $i \in \{1, 2, \dots, k\}$.*

The ID algorithm, called qID utilizes FindQIS to identify as many coefficients as possible in a given model with graph G . It iterates through each connected edge set and attempts to identify it using FindQIS . If it is unable to identify the connected edge set, it then attempts to identify subsets of the connected edge set. After the algorithm has attempted to identify each connected edge set, it again attempts to identify each unidentified connected edge set, since each newly identified coefficient may enable the identification of previously unidentifiable coefficients. This process is repeated until all coefficients have been identified or no new coefficients have been identified in the last iteration. The algorithm is polynomial if the degree of each node in the graph is bounded.

Our algorithm identifies the model depicted in Figure 4b in the following way. First, let us assume that the connected edge sets are arbitrarily ordered, $(\{a\}, \{b, c, f\}, \{d\}, \{e\})$. Now, the first edge to be identified would be a using w_1 as an IV. There is no auxiliary IV set for $\{b, c, f\}$, and we would attempt to find one for its subsets. We find that $\{b\}$ is identified using $\{x\}$ as an IV set with conditioning set $\{w_1\}$. Now, $\{d\}$ is identified using $y^* = y - bx$, and e is identified using t_2^* . In the second iteration, we return to $\{b, c, f\}$ and find that it is now identified using the auxiliary IV set, $\{x, w_1, t_3^*\}$.

Algorithm 1 $qID(G, \Sigma, IDEdges)$

Initialize: EdgeSets \leftarrow all connected edge sets in G
repeat
 for all ES in EdgeSets such that
 $ES \not\subseteq IDEdges$ **do**
 $y \leftarrow He(ES)$
 for all $E \subseteq ES$ such that $E \not\subseteq IDEdges$ **do**
 $(Z, W) \leftarrow FindQIS(G, ES, IDEdges)$
 if $(Z, W) \neq \perp$ **then**
 Identify ES using Z^* as an auxiliary
 instrumental set in $G^{(IDEdges \cap Inc(Z))^+}$
 $IDEdges \leftarrow IDEdges \cup ES$
 end if
 end for
end for
until All coefficients have been identified or no coefficients
have been identified in the last iteration

In contrast, Figure 4b is not identified using simple instrumental sets and auxiliary variables. We cannot identify b without conditioning on w_1 , which means that the only coefficients identified using auxiliary simple instrumental sets is a . Since Chen et al. (2016) showed that any coefficient identified using the generalized half-trek criterion (g-HTC) can be identified using auxiliary variables and simple instrumental sets, we know that qID is able to identify coefficients and models that the g-HT algorithm is not. Moreover, qID will identify any coefficients that are identifiable using auxiliary variables and simple instrumental sets, giving us the following theorem.

Theorem 4. *Given an arbitrary linear causal model, if a set of coefficients is identifiable using the g-HT algorithm, then it is identifiable using qID . Additionally, there are models that are not identified using the g-HT algorithm, but identified using qID .*

5 Deriving Testable Implications using AVs

Theorem 1 also enables us to derive new vanishing partial correlation constraints that can be used to test the model. For example, in Figure 4a, α can be identified using z_1 as an instrument. Once α is identified, we can generate the AV $y^* = y - \alpha x = y - \frac{\sigma(y, z_1)}{\sigma(x, z_1)}x$, and Theorem 1 tells us that the correlation of z_2 and y^* should vanish. As a result, we can test the model specification by verifying that this constraint holds in the data.

Theorem 1 also tells us that the correlation between z_1 and y^* should also vanish. However, upon closer inspection, we find that this implication does not actually constrain the covariance matrix:

$$\begin{aligned} \sigma(z_1, y^*) &= \sigma(z_1, y - \alpha x) \\ &= \sigma(z_1, y) - \frac{\sigma(y, z_1)}{\sigma(x, z_1)}\sigma(z_1, x) = 0. \end{aligned}$$

In other words, our “testable implication” that $\sigma(z_1, y^*) = 0$ is equivalent to stating $\sigma(z_1, y) - \sigma(z_1, y) = 0$ —a tautology!

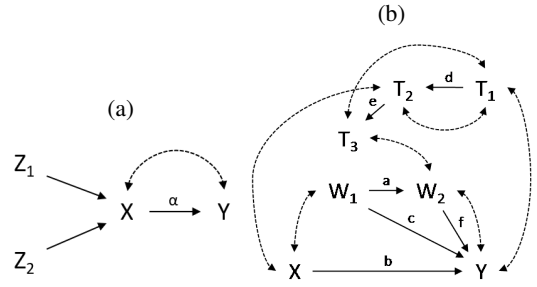


Figure 4: (a) $\sigma(z_2, y^*) = 0$, where $y^* = y - \frac{\sigma(y, z_1)}{\sigma(x, z_1)}$, and, equivalently, α is overidentified using z_1 and z_2 as IVs (b) the model is identified using auxiliary instrumental sets, but not the g-HT algorithm

In contrast,

$$\sigma(z_2, y^*) = \sigma(z_2, y) - \frac{\sigma(z_1, y)}{\sigma(x, z_1)}\sigma(z_2, x) = 0$$

does provide a true testable implication.

Shpitser et al. (2009) noticed a similar phenomenon when deriving dormant independences in non-parametric models, and their explanation applies to conditional independence constraints among AVs as well. The idea is the following: When the model implies that two variables are conditionally independent, it relies on the modeled assumption that there is no edge between those variables. As a result, verifying that the constraint holds in data represents a test that this assumption is valid. However, unlike conditional independence constraints between model variables, conditional independence constraints among AVs rely upon the absence of certain edges in order to identify the coefficients necessary to generate the AV. The key point is that this identification cannot rely on the same lack of edge whose existence we are trying to test!

In the above example, we identified α using z_1 as an IV. $\sigma(z_2, y^*) = 0$ follows from the lack of edge between z_2 and y . However, even if this edge did exist, z^* still equals $z - \frac{\sigma(y, z_1)}{\sigma(x, z_1)}x$. In contrast, $\sigma(z_1, y^*) = 0$ follows from the lack of edge between z_1 and y . The existence of this edge would disallow z_1 as an instrument and $z^* = z - \alpha x \neq z - \frac{\sigma(y, z_1)}{\sigma(x, z_1)}x$.

Another way to derive the constraint $\sigma(z_2, y^*) = 0$ is via overidentification. α can be identified using either z_1 or z_2 and equating the corresponding expressions yields the constraint $\frac{\sigma(y, z_1)}{\sigma(x, z_1)} = \frac{\sigma(y, z_2)}{\sigma(x, z_2)}$, which is clearly equivalent to the previous constraint $\sigma(z_2, y^*) = 0$. In fact, we show (Theorem 6) that whenever a variable z cannot be separated from another variable y , but z^* can be, the resulting AV conditional independence, if it is non-vacuous, is equivalent to an overidentifying constraint that can be derived using quasi-IVs. As a result, all non-vacuous AV conditional independences are captured by overidentifying constraints derived using quasi-IVs!

First, we give a sufficient condition for when a set of edges α is overidentified.

Theorem 5. *Let Z be a quasi-IV set for structural coefficients $\alpha = \{\alpha_1, \dots, \alpha_k\}$ and E be a set of known edges. If there*

exists a node s satisfying the conditions listed below, then α is overidentified and we obtain the constraint .

- (i) $s \notin Z$
- (ii) There exists an unblocked path between s and y including an edge in α
- (iii) There exists a conditioning set W that does not block the path p , such that either:
 - (a) the elements of W are non-descendants of y , and $(s \perp\!\!\!\perp y|W)_{G_{\alpha \cup E_y^-}}$, where $E_y = E \cap \text{Inc}(y)$
 - (b) the elements of W are non-descendants of s and y , and $(s \perp\!\!\!\perp y|W)_{G_{\alpha \cup E_s \cup E_y^-}}$ where $E_s = E \cap \text{Inc}(s)$.

The above theorem can be used to derive an overidentifying constraint for every variable that satisfies (i)-(iii) above. It can also be applied when α is known a priori, yielding a z -overidentifying constraint. In this case, $Z = \emptyset$ would be a quasi-IV set that trivially identifies α .

The following theorem states that non-vacuous AV conditional independence constraints are subsumed by quasi-IV overidentifying and z -overidentifying constraints.

Theorem 6. Let $z^* = z - e_1 t_1 - \dots - e_k t_k$ and suppose there does not exist W such that $(z \perp\!\!\!\perp y|W)_G$. There exists W such that $W \cap \text{De}(z) = \emptyset$ and $(z^* \perp\!\!\!\perp y|W)$ is non-vacuous if and only if y satisfies the conditions of Theorem 5 for $E = \{e_1, \dots, e_k\}$.

The above theorem also applies when y is an AV, called y^* . In this case, we simply replace $(z \perp\!\!\!\perp y|W)_G$ with $(z \perp\!\!\!\perp y^*|W)_{G^{E_y^+}}$, where $E_y \subseteq \text{Inc}(y)$ is a set of edges whose coefficient values are known.

Algorithm 2 uses quasi-IV sets to output overidentifying constraints in a graph given an optional set of identified edges. It uses isEIV, which is a slightly modified version of FindQIS that tests whether w fits the conditions of Theorem 6. Details of isEIV can be found in the Appendix.

Algorithm 2 Finds overidentifying constraints for G

```

function CONSTRAINTFINDER( $G, \Sigma, \text{IDEdges}$ )
  for all  $ES \in \text{Edge Sets of } G$  do
     $(Z, W) \leftarrow \text{FINDQIS}(ES, G, \text{IDEdges})$ 
    if  $(Z, W) \neq \perp$  then
      for all  $w \in V \setminus Z \cup \{He(ES)\}$  do
        if  $\text{ISEIV}(w, ES, G, \text{IDEdges})$  then
          Add constraint  $a_w A^{-1} b = b_w$ 
        end if
      end for
    end if
  end for
end function

```

6 Discussion and Related Work

In this section, we discuss how (single-variable) auxiliary IVs encompass a number of previous identification methods developed in economics (Hausman and Taylor, 1983), computer science (Chan and Kuroki, 2010), and epidemiology (Shardell, 2012).

Hausman and Taylor (1983) showed that if the equation for a given variable, $z = \beta_1 p_1 + \dots + \beta_k p_k + u_z$, is identified, then the error term u_z can be estimated and used as an instrument for other coefficients. In this case, the auxiliary variable $z^* = z - \beta_1 p_1 - \dots - \beta_k p_k$ is equal to the error term u_z . As a result, whenever the error term is estimable and can be used as an IV, we can also generate an auxiliary instrument. However, there are times when only some of the coefficients in an equation are identifiable, and as a result, the error term cannot be used as an instrument, but we can nevertheless generate an auxiliary instrument. As a result, auxiliary IVs strictly subsume error term IVs.

Chan and Kuroki (2010) gave sufficient conditions for when a descendant of x and a descendant of y could be used in analogous manner to IVs to identify the effect of x on y . In the context of AVs, this method is equivalent to generating an auxiliary instrument from the descendant by subtracting the total effect of x on the descendant or the total effect of y on the descendant (depending on whether the variable is a descendant of x or y). In this paper, we generated AVs by subtracting out direct effects, but clearly the work can be extended to subtracting out total effects. The benefit of AVs over these descendant IVs is that they can be generated from a variety of variables, not just descendants of x and y . Additionally, descendants of x or y can generate AVs from other total or direct effects, not just the effect of x or y on the descendant.

The notion of “subtracting out a direct effect” in order to turn a variable into an instrument was also noted by Shardell (2012) when attempting to identify the total effect of x on y . It was noticed that in certain cases, the violation of the independence restriction of a potential instrument z (i.e. z is not independent of the error term of y) could be remedied by identifying, using ordinary least squares regression, and then subtracting out the necessary direct effects on y . AVs generalize and operationalize this notion so that it can be used on arbitrary sets of known coefficient values and be utilized in conjunction with existing graphical methods for identification and enumeration of testable implications.

Additionally, as we have alluded to earlier, the highly algebraic, state-of-the-art g-HTC can also be understood in terms of auxiliary instruments. Identification using the g-HTC is equivalent to identification using auxiliary simple instrumental sets.

In summary, auxiliary instruments are not only the basis for the most general identification algorithm yet devised, but they also unify disparate identification methods under a single framework. Moreover, AVs are directly applicable to the tasks of z -identification and model testing. Finally, they can, in principle, enhance any method for identification, model testing, or other tasks that relies on graphical separation.

7 Conclusion

In this paper, we graphically characterized conditional independence among AVs, allowing us to demonstrate how they can help generalized instrumental sets in the problem of identification. We provided an algorithm that identifies more models than the g-HT algorithm, subsuming the state-of-the-

art for identification in linear models. Additionally, we introduced quasi-IV sets, and constructed an algorithm that utilizes them to attack the problem of z-identification. Finally, we proved that AV conditional independences are subsumed by overidentifying constraints and gave an algorithm for deriving overidentifying constraints.

8 Acknowledgments

We would like to thank Judea Pearl, Mathias Drton, Thomas Richardson, and Luca Weihs for helpful discussions. This research was supported in parts by grants from NSF #IIS-1302448, #IIS-1527490, ONR #N00014-13-1-0153, #N00014-13-1-0153, and DARPA ARO W911NF-16-1-0579.

A Appendix

A.1 Conditional Edge Lemmas

First we give 3 lemmas which are used extensively in the coming proofs. They are referred to as the *Conditional Edge Lemmas*, or *CEL*.

For convenience, we will use a shorthand notation of $\sigma_{xy.W}^G = \sigma(x, y|W)$ in the graph G .¹⁰

Conditional Edge Lemma 1. *Given variables x, y , a conditioning set W , and defining $p_i = Pa(x)_i$, then $\sigma_{xy.W} = \sum_i \alpha_i \sigma_{p_i y.W} + \sigma_{u_x y.W}$, where α_i is the structural parameter for the edge between p_i and x , and u_x is the error term of x .*

Proof. Let $\{w_1, \dots, w_n\} = W$. By definition of conditional covariance,

$$\sigma_{xy.W} = \mathbf{E}[\eta_{x.W} \eta_{y.W}]$$

where $\eta_{x.W}$ is the residual:

$$\eta_{x.W} = x - \sum_i \beta_i w_i$$

with the β_i as regression coefficients. Note that by definition of the residual $\mathbf{E}[w_i \eta_{y.W}] = 0$, i.e. the covariance of a residual with any of its subtracted variables is 0.

$$\begin{aligned} \sigma_{xy.W} &= \mathbf{E}[\eta_{x.W} \eta_{y.W}] = \mathbf{E} \left[\left(x - \sum_i \beta_i w_i \right) \eta_{y.W} \right] \\ &= \mathbf{E}[x \eta_{y.W}] \end{aligned}$$

Expanding the definition of x :

$$\begin{aligned} \mathbf{E}[x \eta_{y.W}] &= \mathbf{E} \left[\left(\sum_i \alpha_i p_i + u_x \right) \eta_{y.W} \right] \\ &= \sum_i \alpha_i \mathbf{E}[p_i \eta_{y.W}] + \mathbf{E}[u_x \eta_{y.W}] \end{aligned}$$

¹⁰Different graphs can have different covariances of the same variables. Since each graph is defined by SEMs, the effect of adding or removing variables to equations (edges) is well-defined in terms of the covariances.

We now subtract the regression coefficients for each variable, since we are subtracting 0 in the expectation (covariance of a residual with its subtracted variables is 0), turning the p_i back into residuals.

$$\begin{aligned} &\sum_i \alpha_i \mathbf{E}[p_i \eta_{y.W}] + \mathbf{E}[u_x \eta_{y.W}] \\ &= \sum_i \alpha_i \mathbf{E}[\eta_{p_i.W} \eta_{y.W}] + \mathbf{E}[\eta_{u_x.W} \eta_{y.W}] \\ &= \sum_i \alpha_i \sigma_{p_i y.W} + \sigma_{u_x y.W} \end{aligned}$$

□

Conditional Edge Lemma 2. *Given a conditional covariance $\sigma_{xy.W}$ in graph G , labeled as $\sigma_{xy.W}^G$, and a set of directed edges E , where G_{E-} is the graph G with edges E removed, if $(W \cup \{x, y\}) \cap Desc(Head(E)) = \emptyset$, then $\sigma_{xy.W}^G = \sigma_{xy.W}^{G_{E-}}$.*

Proof. As done in CEL 1, we directly use the definition of conditional covariance in terms of regression:

$$\sigma_{xy.W} = \mathbf{E}[\eta_{x.W} \eta_{y.W}] \quad \text{where} \quad \eta_{x.W} = x - \sum_i \beta_i w_i$$

β is computed by minimizing the squared residual:

$$\begin{aligned} \mathbf{E}[\eta_{x.W} \eta_{x.W}] &= \mathbf{E} \left[\left(x - \sum_i \beta_i w_i \right)^2 \right] \\ &= \mathbf{E}[x^2] - \sum_i \beta_i \left(2\mathbf{E}[x w_i] - \sum_j \beta_j \mathbf{E}[w_i w_j] \right) \end{aligned}$$

This equation holds in all graphs. We will show that the expectation terms of the equation, and hence the resulting values of β after performing regression are the same in G as they are in G_{E-} .

Since $(W \cup \{x, y\}) \cap Desc(Head(E)) = \emptyset$, we know that x and w_i are both non-descendants of the removed edges in G_{E-} , so the $\mathbf{E}[x^2]$, and all $\mathbf{E}[x w_i]$ and $\mathbf{E}[w_i w_j]$ terms can be directly expanded in terms of their ancestors, which are the same for both G and G_{E-} , and have the same underlying error distribution and covariances¹¹. This means that these terms must be equal in G and G_{E-} .

Another way to reason about this is to use Wright's rules of path analysis. The terms $\mathbf{E}[x w_i]$ can be written in terms of paths between x and w_i . For a path to cross a removed edge, it would need to cross a collider in order to leave the descendants of the edge, and get to the goal node. This means that the valid paths are the same for both graphs, giving the equations used to solve for β identical expectation coefficients.

We can now expand out the value of $\sigma_{xy.W}$ the same way in both graphs:

¹¹we are working in DAGs - non-recurrent models

$$\begin{aligned}\sigma_{xy \cdot W} &= \mathbf{E}[\eta_{x \cdot W} \eta_{y \cdot W}] = \mathbf{E}[\eta_{x \cdot W} y] \\ &= \mathbf{E}\left[x - \sum_i \beta_i w_i\right] y = \mathbf{E}[xy] - \sum_i \beta_i \mathbf{E}[w_i y]\end{aligned}$$

We have showed that β are the same in both graphs, and we use the same reasoning to conclude that $\mathbf{E}[w_i y]$ and $\mathbf{E}[xy]$ must be equal in G and G_{E-} . Therefore, since all terms in the equation are the same in both graphs, $\sigma_{xy \cdot W}^G = \sigma_{xy \cdot W}^{G_{E-}}$. \square

Conditional Edge Lemma 3. *Given a conditional error covariance $\sigma_{u_x y \cdot W}^G$, and a set of directed edges E , if $(W \cup \{y\}) \cap \text{Desc}(\text{Head}(E)) = \emptyset$, then $\sigma_{u_x y \cdot W}^G = \sigma_{u_x y \cdot W}^{G_{E-}}$.*

The main difference between this and CEL 2, is that we operate on u_x (the error term of x), which allows x to be a descendant of $\text{Head}(E)$.

Proof. We proceed in the same fashion as in CEL 2. By the definition of conditional error covariance:

$$\begin{aligned}\sigma_{u_x y \cdot W} &= \mathbf{E}[\eta_{u_x \cdot W} \eta_{y \cdot W}] = \mathbf{E}[u_x \eta_{y \cdot W}] \\ &= \mathbf{E}\left[u_x \left(y - \sum_i \beta_i w_i\right)\right] = \mathbf{E}[u_x y] - \sum_i \beta_i \mathbf{E}[u_x w_i]\end{aligned}$$

Using the reasoning from CEL 2, we know that β_i are the same for G and G_{E-} . Once again, expanding y and w_i to their ancestors, which have no edges removed, we get the same distributions for both graphs, meaning that the expectations are also equal.

This can also be seen intuitively in terms of Wright's rules when x is not an ancestor of y . In that case, $\mathbf{E}[u_x y]$ represents all paths from x to y starting with a bidirected edge (half-treks). If such a path were to be different in the two graphs, it would need to cross a deleted edge. But to do that, it would have to cross a collider. If x is an ancestor of y , then we will additionally have an $\mathbf{E}[u_x u_x]$ term in our expansion, which is the same for both graphs. \square

A.2 Auxiliary and Quasi-Instrumental Sets

Theorem 1. *Given a linear SEM with graph G , where $E \subseteq \text{Inc}(z)$ is a set of edges whose coefficient values are known, if $W \cup \{y\}$ does not contain descendants of z , then $(z^* \perp\!\!\!\perp y|W)_{G^{E+}}$ if and only if $(z \perp\!\!\!\perp y|W)_{G_{E-}}$. Furthermore, $\sigma_{z^* y \cdot W}^{G^{E+}} = \sigma_{zy \cdot W}^{G_{E-}}$.*

Proof. Let e_1, \dots, e_k be the coefficients of E , $\{t_1, t_2, \dots, t_k\} = \text{Ta}(E)$, let $\{t'_1, t'_2, \dots, t'_l\}$ be parents of z that are not in $\text{Ta}(E)$, and $\{w_1, \dots, w_n\} = W$. Remember that by definition of auxiliary variable:

$$z^* = z - \sum_{i=1}^k e_i t_i$$

In linear SEMs, $(z^* \perp\!\!\!\perp y|W)$ iff $\sigma_{z^* y \cdot W} = 0$ (in the graph G^{E+} , the E -augmented graph). Using the Conditional Edge Lemmas:

$$\begin{aligned}\sigma_{z^* y \cdot W}^{G^{E+}} &= \sigma_{zy \cdot W} - \sum_{i=1}^k e_i \sigma_{t_i y \cdot W} \\ &= \sum_{i=1}^k e_i \sigma_{t_i y \cdot W} + \sum_{i=1}^l e'_i \sigma_{t'_i y \cdot W} - \sum_{i=1}^k e_i \sigma_{t_i y \cdot W} + \sigma_{u_z y \cdot W} \\ &= \sum_{i=1}^l e'_i \sigma_{t'_i y \cdot W} + \sigma_{u_z y \cdot W} = \sigma_{zy \cdot W}^{G_{E-}}\end{aligned}$$

In the last 2 steps, we used CEL 2 and 3 to switch this equation from G^{E+} to G_{E-} (since $\sigma_{t'_i y \cdot W}^{G^{E+}} = \sigma_{t'_i y \cdot W}^{G_{E-}}$ and $\sigma_{u_z y \cdot W}^{G^{E+}} = \sigma_{u_z y \cdot W}^{G_{E-}}$), and CEL 1 to realize that the resulting sum is $\sigma_{zy \cdot W}$ in G_{E-} . \square

Supplemental Definition 1. *Given a linear SEM with graph G , a set E_Z of known coefficients, and a set of structural coefficients $\alpha = \{\alpha_1, \alpha_2, \dots, \alpha_k\}$, the set $Z = \{z_1, \dots, z_k\}$ generates an auxiliary instrumental set if there exist triples $(z_1, W_1, p_1), \dots, (z_j, W_j, p_k)$ such that:*

1. For $i = 1, \dots, k$, either:
 - (a) the elements of W_i are non-descendants of y , and $(z_i \perp\!\!\!\perp y|W_i)_{G_{E-}}$ where G_{E-} is the graph obtained by deleting the edges E from G .
 - (b) the elements of W_i are non-descendants of z_i and y , and $(z_i \perp\!\!\!\perp y|W_i)_{G_{E \cup E_{z_i}}}$ where $G_{E \cup E_{z_i}}$ is the graph obtained by deleting the edges $E, E_Z \cap (\text{Inc}(z_i))$ from G .
2. for $i = 1, \dots, k$, p_i is an unblocked path between z_i and y , not blocked by W_i , where $x_i = \text{He}(\alpha_i)$,
3. the set of paths $\{p_1, \dots, p_k\}$ has no sided intersection

Supplemental Theorem 1. *If there exists an auxiliary instrumental set for structural coefficients $\{\alpha_1, \alpha_2, \dots, \alpha_k\}$, then the coefficients are identifiable.*

Proof. This proof is a modification of Brito and Pearl (2002a)'s proof of instrumental sets. The modifications span multiple lemmas, therefore the full proof is given as appendix B of this document (below). \square

Theorem 3. *If Z^* is a quasi-instrumental set for E , then the coefficients E are identifiable.*

Proof. Suppose we have a quasi-instrumental set for $E = \{e_1, \dots, e_k\}$ with $Z^* = \{z_1, \dots, z_k\}$ (z_i is referring to the auxiliary variable itself rather than its generator). We know that this set is solvable in the graph $G_{E_{z_i}}$, where the graph is obtained by deleting the edges $T = E_Z \cap \text{Inc}(y)$ from G , since it is an auxiliary instrumental set for the graph.

Let the parameters connecting $t \in T$ to y be γ . Let T' be all incident edges to y that are not in T or E . That is, $T' = \text{Inc}(y) \setminus (E \cup T)$ (and let the associated structural parameters be γ'). Finally, let X be $\text{Tail}(E)$.

We will show that there exists a solution by explicitly constructing the linear equations to be solved for the parameters. For each z_i , we generate an equation:

$$\begin{aligned}\sigma_{z_i y^* . W_i} &= \sigma_{z_i y . W_i} - \sum_j \gamma_j \sigma_{z_i t_j . W_i} \\ &= \sum_j e_j \sigma_{z_i x_j . W_i} + \sum_j \gamma'_j \sigma_{z_i t'_j . W_i} + \sigma_{z_i u_y . W_i}\end{aligned}$$

We will use the Conditional Edge Lemmas to move the last two terms into the graph $G_{E-\cup E_y}$, where these terms are equal to $\sigma_{y z_i . W_i}^{G_{E-\cup E_y}}$. We notice that the second term in the resulting equation must be 0, since by definition of quasi-IV ($z \perp\!\!\!\perp y | W$) $_{G_{E-\cup E_y}}$

$$\begin{aligned}\sigma_{z_i y^* . W_i} &= \sum_j e_j \sigma_{z_i x_j . W_i} + \sum_j \gamma'_j \sigma_{z_i t'_j . W_i} + \sigma_{z_i u_y . W_i} \\ &= \sum_j e_j \sigma_{z_i x_j . W_i} + \sigma_{y z_i . W_i}^{G_{E-\cup E_y}} \\ &= \sum_j e_j \sigma_{z_i x_j . W_i}\end{aligned}$$

We now have a system of linear equations, one for each z_i , in terms of the e_i . The system is in the form $Ae = b$. The A matrix is full rank, because by the Conditional Edge Lemmas, all terms in the matrix are the same as their counterparts in $G_{E-\cup E_y}$. We know that if we find a quasi-instrumental set, then there exists at least one quasi-instrumental set Z^* which makes this matrix full rank. We proved the existence of such a set in supplementary theorem 1. That is, we showed that if one auxiliary set exists, we can always construct another for E , for which the above matrix is full rank, and thus invertible. For details, see proof of Supplemental Theorem 1. \square

Corollary 1. *Given a linear SEM with graph G , z^* is a quasi-IV for α given W if W does not contain any descendants of z , and z is an IV for α given W in $G_{E_z \cup E_y -}$, where $E_z \subseteq \text{Inc}(z)$ and $E_y \subseteq \text{Inc}(y)$ are sets of edges whose coefficient values are known.*

Proof. Let IV-(i), IV-(ii), and IV-(iii) denote conditions (i)-(iii) of Lemma 1 in Pearl (2011) and let α be the coefficient of edge (x, y) . We need to show that IV-(i), IV-(ii), and IV-(iii) hold in G^{E+} . Since z is an IV for α given W in G_E , it must be the case that z^* satisfies IV-(i) and IV-(iii) in G^{E+} . Now, it remains to be shown that $(z^* \perp\!\!\!\perp y | W)_{G_\alpha^{E+}}$. Theorem 1 tells us that if $(z \perp\!\!\!\perp y | W)_{G_{E \cup \{\alpha\}}}$ and $W \cup \{y\}$ does not contain descendants of z in $G_{E \cup \{\alpha\}}$, then $(z^* \perp\!\!\!\perp y | W)_{G_\alpha^{E+}}$. By assumption, W does not contain any descendants of z . y also cannot be a descendant of z in $G_{E \cup \{\alpha\}}$. If y were a descendant of z , then it would not be possible to block the path from z to y using W , which does not contain any descendants of z . \square

Theorem 4. *Given an arbitrary linear causal model, if a set of coefficients is identifiable using the g-HT algorithm, then*

it is identifiable using qID. Additionally, there are models that are not identified using the g-HT algorithm, but identified using qID.

Proof. Proved in the paragraph preceding theorem statement in paper \square

Theorem 5. *Let Z be a quasi-IV set for structural coefficients $\alpha = \{\alpha_1, \dots, \alpha_k\}$ and E be a set of known edges. If there exists a node s satisfying the conditions listed below, then α is overidentified.*

1. $s \notin Z$
2. *There exists an unblocked path between s and y including an edge in α*
3. *There exists a conditioning set W that does not block the path p , such that either:*
 - (a) *the elements of W are non-descendants of y , and $(s \perp\!\!\!\perp y | W)_{G_{\alpha \cup E_y -}}$, where $E_y = E \cap \text{Inc}(y)$*
 - (b) *the elements of W are non-descendants of s and y , and $(s \perp\!\!\!\perp y | W)_{G_{\alpha \cup E_s \cup E_y -}}$ where $E_s = E \cap \text{Inc}(s)$.*

Proof. In the proof of theorem 3, we generated a full-rank set of linear equations, where each equation had the form:

$$\sigma_{z_i y^* . W_i} = \sum_j e_j \sigma_{z_i x_j . W_i}$$

We can generate a set of linear equations of the form $Ae = b$, using the above.

Similarly, we can use the parameter s to generate another single equation in the given form: $a_s e = b_s$. Now, if Z_E is a full auxiliary set, then A is invertible, so we get $e = A^{-1}b$, giving us the overidentifying constraint $a_s A^{-1}b = b_s$. \square

Theorem 6. *Let $z^* = z - e_1 t_1 - \dots - e_k t_k$ and suppose there does not exist W such that $(z \perp\!\!\!\perp y | W)_G$. There exists W such that $W \cap \text{De}(z) = \emptyset$ and $(z^* \perp\!\!\!\perp y | W)$ is non-vacuous if and only if y satisfies the conditions of Theorem 5 for $E = \{e_1, \dots, e_k\}$.*

Proof. (\implies) First, we show that y satisfies (ii) and (iii) of Theorem 5. Since $z \not\perp\!\!\!\perp y | W$ but $z^* \perp\!\!\!\perp y | W$ there must exist a path from y to z that goes through E and (ii) is satisfied. Next, $(z^* \perp\!\!\!\perp y | W)$ implies that $(z \perp\!\!\!\perp y | W)_{G_{E-}}$ so (iii) is satisfied.

Next, we show that there exists $T = \{t_1, \dots, t_k\}$, $y \notin T$, such that T is a quasi-IV set for E so that (i) is satisfied. Since $(z^* \perp\!\!\!\perp y | W)$ is not vacuous, E is identified in G' , the graph where a directed edge from y to z , called e_{yz} , is added. As a result, there exists T such that $y \notin T$ and $T \cup \{y\}$ is a quasi-IV set for $E \cup \{e_{yz}\}$. It follows that T is a quasi-IV set for E .

(\impliedby) Let T be the quasi-IV set for E that does not include y . (iii) implies that there exists W such that $(y \perp\!\!\!\perp z | W)_{G_{E-}}$, and, since E is identifiable using T , $(z^* \perp\!\!\!\perp y | W)$. Finally, this independence cannot be vacuous since $T \cup \{y\}$ is a quasi-IV set for $E \cup \{e_{yz}\}$ in G' . \square

A.3 Identification and z-Identification Algorithm

Two algorithms are given for finding Quasi-Instrumental Sets. The first version does not consider IVs that are conditioned on descendants of z , whereas the second version is more computationally expensive (still polynomial if k is bounded), but is able to find any quasi-instrumental set if such exists.

In *FindQIS*, we make extensive use of *TestQIS*, which is a modification of *TestGeneralIVs*(G, X, Y, Z) from van der Zander and Liskiewicz (2016). Our version has 2 extra arguments, and replaces the first 4 lines of *TestGeneralIVs* such that we can search for both auxiliary instruments ($Aux = 1$) and standard instrumental variables ($Aux = 0$).

Algorithm 3 Modified version of *TestGeneralIVs* from van der Zander and Liskiewicz (2016) for use with *findAuxIS*

```

function TESTQIS( $G, X, Y, Z, IDEdges, Aux$ )
  for  $i$  in  $1, \dots, |Z|$  do
    if  $Aux_i == 1$  then
       $W_i \leftarrow$  a nearest separator for  $(Y, Z_i)$ 
      in  $G_{E \cup E_{z_i} \cup E_y}$ , where  $E_{z_i}$ 
      is  $IDEdges \cap Inc(z_i)$ 
      if  $W_i = \perp \vee (W_i \cap De(Y)) \neq \emptyset \vee (W_i \cap$ 
       $De(z_i)) \neq \emptyset$  then
        return  $\perp$ 
      end if
    else
       $W_i \leftarrow$  a nearest separator for  $(Y, Z_i)$  in  $G_{E \cup E_y}$ 
      if  $W_i = \perp \vee (W_i \cap De(Y)) \neq \emptyset$  then
        return  $\perp$ 
      end if
    end if
  end for
  continue algorithm TestGeneralIVs starting
  from second for loop.
  Instead of returning False, return  $\perp$ ,
  and instead of returning True, return  $W$ .
end function

```

Algorithm 4 Finds a quasi-instrumental set (without conditioning on descendants in IVs)

```

function FINDQISBASIC( $E, G, IDEdges$ )
  for all  $Z \subset V \setminus \{y\}$  of size  $|E|$  do
     $W \leftarrow$  TESTQIS( $G, Ta(E), Head(E), Z, 1$ )
    if  $W \neq \perp$  then
      return  $(Z, W)$ 
    end if
  end for
  return  $\perp$ 
end function

```

Algorithm 5 Finds a quasi-instrumental set for E in G , given a set $IDEdges$ of identified edges.

```

function FINDQIS( $E, G, IDEdges$ )
  for all  $Z \subset V \setminus \{y\}$  of size  $|E|$  do
    for all  $Aux \in \{0, 1\}^{|E|}$  do
       $W \leftarrow$  TESTQIS( $G, Ta(E), Head(E), Z, Aux$ )
      if  $W \neq \perp$  then
        return  $(Z, W)$ 
      end if
    end for
  end for
  return  $\perp$ 
end function

```

Algorithm 6 Tests whether w fits the conditions of theorem 6

```

function ISEIV( $w, E, G, IDEdges$ )
  Let  $G'$  be the graph  $G$  modified such that  $E$  are
  removed, and each node in  $Tail(E)$  has an edge
  added to a newly created node  $n$ , which has an
  edge to  $Head(E)$ 
  for all  $Aux \in \{0, 1\}$  do
     $W \leftarrow$  TESTQIS( $G', \{n\}, Head(E), \{w\}, Aux$ )
    if  $W \neq \perp$  then
      return  $(Z, W)$ 
    end if
  end for
  return  $\perp$ 
end function

```

The function *IsEIV*, is a slight modification of *FindQIS* that makes the subset a full auxiliary set in a graph modified so that the full set of E has directed edges to a single node, instead of y , so that this node can be a new set E' of size 1.

B Proof of Supplemental Theorem 1

We build upon the proof given in Brito and Pearl (2002a) to show that auxiliary instrumental sets are identifiable.

B.1 Generalized Instrumental Sets

We will use the definition of generalized instrumental set directly from Brito and Pearl (2002a)'s paper.

Definition 7. *The set Z is said to be an instrumental set relative to X and Y if we can find triples $(Z_1, W_1, p_1), \dots, (Z_n, W_n, p_n)$ such that for $i = 1, \dots, n$*

1. Z_i and the elements of W_i are non-descendants of Y ; and p_i is an unblocked path between Z_i and Y including edge $X_i \rightarrow Y$
2. Let \bar{G} be the causal graph obtained from G by deleting edges $X_1 \rightarrow Y, X_n \rightarrow Y$. Then W_i d -separates Z_i from Y in \bar{G} , but W_i does not block path p_i

3. For $1 \leq i \leq j \leq n$, Z_j does not appear in path p_i , and, if paths p_i and p_j have a common variable V , then both $p_i[V \sim Y]$ and $p_j[Z_j \sim V]$ point to V .

The third property is written here in the same way it is written in Brito and Pearl (2002a). We used p_i and p_j do not have any sided intersection instead. The two methods for writing the property are equivalent, meaning that there exists a set satisfying the Brito and Pearl (2002a) definition iff there exists a set satisfying our definition (note that the two sets might be different). This is proved in Appendix C of this document.

B.2 Auxiliary Instrumental Sets

We perform an equivalent translation to the definition of Auxiliary Instrumental Set:

Definition 8. Given a linear SEM with graph G , a set E_Z of known coefficients, and a set of structural coefficients $\alpha = \{\alpha_1, \alpha_2, \dots, \alpha_k\}$, the set $Z = \{z_1, \dots, z_k\}$ generates an auxiliary instrumental set if there exist triples $(z_1, W_1, p_1), \dots, (z_j, W_j, p_k)$ such that:

1. For $i = 1, \dots, k$, either:
 - (a) the elements of W_i are non-descendants of y , and $(z_i \perp\!\!\!\perp y | W_i)_{G_E}$ where G_E is the graph obtained by deleting the edges E from G .
 - (b) the elements of W_i are non-descendants of z_i and y , and $(z_i \perp\!\!\!\perp y | W_i)_{G_{E \cup E_{z_i}}}$ where $G_{E \cup E_{z_i}}$ is the graph obtained by deleting the edges E , $E_Z \cap (Inc(z_i))$ from G .
2. for $i = 1, \dots, k$, p_i is an unblocked path between z_i and y , not blocked by W_i , including the edge (x_i, y)
3. For $1 \leq i \leq j \leq n$, Z_j, Z_j' does not appear in path p_i , and, if paths p_i and p_j have a common variable V , then both $p_i[V \sim Y]$ and $p_j[Z_j \sim V]$ point to V .

B.3 Auxiliary Sets generate Generalized Instrumental Sets

Lemma 1. If there exists an auxiliary instrumental set for structural coefficients $\{\alpha_1, \alpha_2, \dots, \alpha_k\}$, then there exists a generalized instrumental set for the coefficients in G^{E+} .

Proof. We will denote conditions 1 through 3 of Supplemental Definition 8 as AIV 1-3, respectively. We will denote the conditions of Definition 7 as GIV 1-3. This proof will proceed by showing that we can generate a generalized instrumental set in G^{E+} using the auxiliary set.

We have defined G^{E+} as the graph where all possible auxiliary variables have been added. For each z_i in Z :

1. if z_i satisfies AIV 1a, then $(z_i \perp\!\!\!\perp y | W)_{G^{E+}}$, because the added node z_i^* is a collider for any possible paths going through it. If z_i satisfies AIV 1b, then $(z_i^* \perp\!\!\!\perp y | W)_{G^{E+}}$ using Theorem 1. Therefore, GIV 1 is satisfied.
2. If AIV 2 is satisfied, then GIV 2 follows directly if AIV 1a is satisfied. If AIV 1b is satisfied, we can extend the path from AIV 2 with the edge $z_i^* \leftarrow z_i$. Since z_i^* is unblocked, this new path will satisfy GIV 2.

3. If AIV 3 is satisfied, then the paths (p_i) constructed in the previous part will not have sided intersection. We might have added the edge $z_i^* \leftarrow z_i$ which makes z_i in $Left(p_i)$, but the original z_i was in $Left(p_i)$ already by the definition of $Left$. Furthermore, z_i^* is a collider, so it could not be part of any other variable's path. This means GIV 3 is satisfied.

Since all of the conditions necessary for definition 7 are satisfied, we have constructed a generalized instrumental set for G^{E+} . \square

B.4 Identifiability of Generalized IVs does NOT imply ID of Aux IVs

In generalized IVs, it is assumed that all edges in the graph have independent structural parameters. When using auxiliary variables, the edges incoming to the auxiliary variable are repeating the structural parameters found elsewhere in the graph. This invalidates the assumption of independence implicit in Definition 7.

Furthermore, it turns out that in proving the identifiability of coefficients from a generalized instrumental set, Brito and Pearl (2002a) generated another instrumental set, with a special property. They argued that this new set still satisfied the conditions of Definition 7. With auxiliary variables, it is not clear that it is possible to modify the auxiliary set, since the independence properties of the variables are different, since Z^* has coefficients cancel only after subtracting the auxiliary paths.

We will show that Brito and Pearl (2002a)'s proof can be modified to show identifiability in auxiliary instrumental sets.

Preliminaries

First, we will quickly review the relevant portions of the proof of generalized IVs.

Lemma 2. (Partial Correlation Lemma, Brito and Pearl (2002a)) The partial correlation $\rho_{12.3\dots n}$ can be expressed as the ratio:

$$\rho_{12.3\dots n} = \frac{\phi(1, 2, \dots, n)}{\psi(1, 3, \dots, n)\psi(2, 3, \dots, n)}$$

where ϕ and ψ are functions satisfying the following conditions:

1. $\phi(1, 2, \dots, n) = \phi(2, 1, \dots, n)$
2. $\phi(1, 2, \dots, n)$ is linear on correlations $\rho_{12}, \rho_{32}, \dots, \rho_{n2}$ with no constant term
3. The coefficients of $\rho_{12}, \rho_{32}, \dots, \rho_{n2}$ in $\phi(1, 2, \dots, n)$ are polynomials on the correlations among Z, W_i, \dots . Furthermore, the coefficient of ρ_{12} has its constant term = 1, and the coefficients of $\rho_{32}, \dots, \rho_{n2}$ are linear on the correlations $\rho_{13}, \rho_{14}, \dots, \rho_{1n}$ with no constant term
4. $(\psi(i_1, \dots, i_{n-1}))^2$ is a polynomial on the correlations among variables $Y_{i_1}, \dots, Y_{i_{n-1}}$ with constant term = 1.

With this lemma in hand, we will outline how Brito and Pearl (2002a) showed that IVs are identifiable by restating the lemmas, and giving 2 sentence descriptions of how they were proved.

Lemma 3. (Lemma 2, Brito and Pearl (2002a)) *WLOG, we may assume that for $1 \leq i < j \leq n$, paths p_i and p_j do not have any common variable other than (possibly) Z_i .*

Proof. (Outline) Suppose not. That is, suppose that paths p_i and p_j have a variable in common other than Z_i . Call this variable V . We can now generate a new instrumental set using V instead of Z_i . That is, if there exists a common variable, we can generate a new instrumental set, where this variable is Z_i . This new instrumental set conforms to the definition 7. This is proved by showing that since Z_i is independent of Y given W_i , V must also be independent, since there is a directed, unblocked, path from V to Z_i . \square

Lemma 4. *For all $1 \leq i \leq n$, there exists no unblocked path between Z_i and Y , different from p_i , which includes edge $X_i \rightarrow Y$, and is composed only of edges from p_1, \dots, p_i .*

Proof. (Outline) By contradiction - suppose such a path exists, then since it is different from p_i , it must contain edges from p_1, \dots, p_{i-1} . But all such paths that intersect with p_1 will do so at a collider. \square

Lemma 5. *For all $1 \leq i \leq n$, there exists no unblocked path between Z_i and some W_i , composed only of edges from p_1, \dots, p_i .*

Lemma 6. *For all $1 \leq i \leq n$, there exists no unblocked path between Z_i and Y , including edge $X_j \rightarrow Y$, with $j < i$, composed only of edges from p_1, \dots, p_i .*

These two lemmas use the same proof method as lemma 4, and the proofs are omitted. Using these 3 lemmas, Brito and Pearl (2002a) proved that the determinant of the linear system is a non-trivial polynomial, whose zeros have lebesgue measure zero.

Proof Modification for Auxiliary Variables

The above lemmas are the only thing which needs to be modified to work with Auxiliary Variables. Lemma 3 needs to be modified to take into account the fact that Auxiliary Variables have different independence properties, whereas lemma 4 and its siblings need to take into account that edges are repeated in our graph.

Lemma 7. *WLOG, we may assume that for $1 \leq i < j \leq n$, paths p_i and p_j do not have any common variable other than (possibly) Z_i or Z'_i (parent of Z_i if it is an auxiliary variable).*

Proof. Assume that paths p_i and p_j have some variables in common, different from Z_i (which might be an auxiliary variable). Let V be the closest variable to X_i in path p_i which also belongs to path p_j . We show that after replacing (Z_i, W_i, p_i) with $(V, W_i, p_i[V \sim Y])$, definition 8 still holds.

From (3), changed to be in the format of GLVs, the subpath $p_i[V \sim Y]$ must point to V . Since p_i is unblocked, subpath $p_i[Z_i \sim V]$ must be a directed path from V to Z_i . Furthermore, if Z_i is an auxiliary variable, p_i did not cross any of the subtracted edges, since the path was found in a graph with these edges removed.

At this point, if the variable Z_i is not an auxiliary variable, the 3 conditions hold:

1. (a) is satisfied, since $p_i[Z_i \sim V]$ is a directed path from V to Z_i , so if V is descendant of y then Z_i is a descendant of y . Similarly, if $(v \perp\!\!\!\perp y|W_i)_{G_E}$, then $(z_i \perp\!\!\!\perp y|W_i)_{G_E}$, because if W_i does not d-separate V from y , then since W_i are not blocking p_i , we can generate a path from Z_i to y through V .
2. Since the path from V to Y is a subpath of the path $Z_i \sim Y$, the path is unblocked.
3. The path from Z_i to y must have $V \in \text{Left}$, since $p_i[Z_i \sim V]$ is a directed path. Therefore, the new path has no sided intersection with any of the other paths in the set.

If Z_i is an auxiliary variable, we will call its parent Z'_i . Conditions 2 and 3 follow using the same proof as given for non-AVs above. The first condition, however, requires more care. The case of $V = Z'_i$ is permitted by assumption.

Suppose $V \neq Z'_i$. That means that the path $p_i[Z_i \sim V]$ goes through one of Z'_i 's incoming edges (and does not go through the auxiliary edges). This path exists in the graph G_{E-} . If V is descendant of y then Z_i is a descendant of y , since the directed path p_i does not get cut in G_{E-} . Similarly, suppose $(v \perp\!\!\!\perp y|W_i)_{G_E}$, then using the Conditional Edge Lemma 2, $(v \perp\!\!\!\perp y|W_i)_{G_{E-}}$. Since there is a directed, unblocked path from v to Z'_i , $(Z'_i \perp\!\!\!\perp y|W_i)_{G_{E-}}$, so using Theorem 1, $(Z_i \perp\!\!\!\perp y|W_i)_{G_{E+}}$ - a contradiction. Therefore $(v \perp\!\!\!\perp y|W_i)_{G_E}$, so v satisfies (a). \square

For the next proof, we will assume that the conditions in lemma B.4 hold.

Lemma 8. *For all $1 \leq i \leq n$, there exists no unblocked path between Z_i and Y , different from p_i , which includes edge $X_i \rightarrow Y$ and is composed only by edges from p_1, \dots, p_i .*

Proof. Let p be an unblocked path between Z_i and Y , different from p_i , and assume that p is composed only by edges from p_1, \dots, p_i . According to the ordering condition, if Z_i or Z'_i appears in some path p_j , with $j \neq i$, then $j > i$. Therefore, p must start at Z_i , and take a non-auxiliary edge from Z'_i . Since p is different from p_i , it must contain at least one edge from p_1, \dots, p_{i-1} . Let (v_1, V_2) denote the first edge in p which does not belong to p_i . From lemma , it follows that V_1 must be a z_k or z'_k for some $k < i$, and the subpath $p_i[Z_i \sim V_1]$ and (V_1, V_2) must point to V_1 . This implies that p is blocked by V_1 (collider), a contradiction. \square

Using the same proof, we also get:

Lemma 9. *For all $1 \leq i \leq n$, there exists no unblocked path between Z_i and some W_i , composed only of edges from p_1, \dots, p_i .*

Lemma 10. *For all $1 \leq i \leq n$, there exists no unblocked path between Z_i and Y , including edge $X_j \rightarrow Y$, with $j < i$, composed only of edges from p_1, \dots, p_i .*

To finish the proof, we add a comment about auxiliary variables to Brito's Lemma 7:

Lemma 11. *The coefficients of edges incident to y are 0 unless they are part of the instrumental set.*

Proof. Using CEL1, we can see that the coefficients are $\sigma_{z p_i, W}$. But these are the same in graph G and G_{E-} by CEL 2. If the coefficient were non-zero in G_{E-} , then $\sigma_{z y, W}$ would be non-zero by d-separation (there is a directed edge from each p_i to y), meaning that conditional independence would be violated. \square

This completes the necessary proof modifications. We were able to sidestep issues of same-value structural parameters by ensuring that all intersections that might move across the auxiliary edges happen with $i < j$, and are not relevant to the proof.

C Equivalence of IV Definitions

For convenience, Definition 7 is restated here in its original (theorem) form:

Theorem 7. (Brito and Pearl, 2002a) *Given a linear model with graph G , the coefficients for a set of edges $E = \{(x_1, y), \dots, (x_k, y)\}$ are identified if there exists triplets $(z_1, W_1, p_1), \dots, (z_k, W_k, p_k)$ such that for $i = 1, \dots, k$,*

1. $(z_i \perp\!\!\!\perp y | W_i)_{G_{E-}}$, where W does not contain any descendants of y and G_{E-} is the graph obtained by deleting the edges, E from G ,
2. p_i is a path between z_i and x_i that is not blocked by W_i , and
3. if $1 \leq i < j \leq n$ the variable z_j does not appear in path p_i ; and, if paths p_i and p_j have a common variable V , then both $p_i[V \sim Y]$ and $p_j[Z_j \sim V]$ point to V .

If the above conditions are satisfied, we say that Z is a generalized instrumental set for E or simply an instrumental set for E .¹²

We will show that the third condition in this theorem can be replaced with an assertion that the paths have no sided intersection. That is, the following theorem is equivalent:

Theorem 8. *Given a linear model with graph G , the coefficients for a set of edges $E = \{(x_1, y), \dots, (x_k, y)\}$ are identified if there exists triplets $(z_1, W_1, p_1), \dots, (z_k, W_k, p_k)$ such that for $i = 1, \dots, k$,*

1. $(z_i \perp\!\!\!\perp y | W_i)_{G_{E-}}$, where W does not contain any descendants of y and G_{E-} is the graph obtained by deleting the edges, E from G ,
2. p_i is a path between z_i and x_i that is not blocked by W_i , and
3. the set of paths, $\{p_1, \dots, p_k\}$ has no sided intersection.

We will perform several reversible steps to show that whenever an instrumental set of one type exists, a set of the other must also exist.

Lemma 12. *There exist triples satisfying the conditions of theorem 8, iff there exist triples satisfying the theorem with condition 3 replaced with: the set of paths $\{p_1, p_2, \dots, p_k\}$ has no sided intersection, and furthermore, the paths are all half-treks*

¹²Note that when $k = 1$, z_1 is an IV for (x_1, y) . Further, if $z_1 = x_1$, then x_1 satisfies the single-door criterion for (x_1, y) .

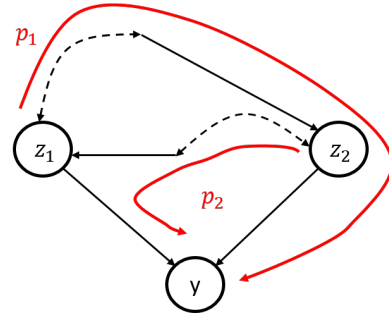


Figure 5: The structure of intersecting paths in lemma 13

Proof. \Leftarrow follows directly, since any set of triples satisfying lemma 12 automatically satisfies theorem 8.

\Rightarrow Suppose we have a set of triples satisfying theorem 8. Consider the set of triples where the i th triple (z_i, W_i, p_i) from theorem 8 is replaced with (z'_i, W_i, p'_i) . We define z'_i as the last variable in $Left(p_i)$ from z_i along p_i ¹³. p'_i is defined as the subpath from z'_i to y ($p_i[z'_i \sim x_i]$).

We now show that this new set of triples satisfies the definition of lemma 12.

1. Suppose $(z'_i \not\perp\!\!\!\perp y | W_i)_{G_{E-}}$. This means that z_i and y are not d-separated given W_i , and as such there exists a path from y to z_i . But there is a directed path from z'_i to z_i , which is also unblocked by W_i . Combining those two paths gives a path between y and z_i , meaning $(z_i \not\perp\!\!\!\perp y | W_i)_{G_{E-}}$, a contradiction.
2. Since p'_i is a subpath of $p_i[z'_i \sim x_i]$, it is a path between z'_i and x_i that is not blocked by W_i .
3. By the definition of p'_i , all of the paths are half-treks. Furthermore, since $\{p_1, p_2, \dots, p_k\}$ had no sided intersection, and $\{p'_1, p'_2, \dots, p'_k\}$ are subpaths of these original paths, and by the fact that z'_i must have already been in $Left(p_i)$, we have $Right(p'_i) = Right(p_i)$, and $Left(p'_i) \subseteq Left(p_i)$ for all i . Therefore, $\{p'_1, p'_2, \dots, p'_k\}$ must not have sided intersection, since if it did, $\{p_1, p_2, \dots, p_k\}$ would have also had this intersection. \square

Corollary 2. *If there exist triples satisfying lemma 12, then the set of paths $\{p_1, \dots, p_k\}$ can only intersect at z_1, \dots, z_k , where z_i is the instrumental variable.*

Proof. If two paths have no sided intersection, then any node that is in both paths must be in $Right$ of one path, and in $Left$ of the other. Since each path p_i is a half-trek, the only variable in $Left$ is z_i , with the rest of the variables in $Right$. Thus any intersection must happen at z_i , the instrumental variable. \square

Lemma 13. *There exist triples satisfying lemma 12 iff there exist triples satisfying the lemma AND $\forall z_i, z_j$, if z_j is on path p_i , then z_i is not on path p_j .*

¹³Remember that since p_i is an unblocked path from z_i to an incoming edge of y , it is a trek starting with one or more nodes in $Left$, and ending with nodes in $Right$

Proof. Using lemma 12, we can generate a set of triples where all paths are half-treks. Suppose that $\exists i, j$ s.t. z_i is on path p_j and z_j is on path p_i . Since p_i is a half-trek, z_i is the only node in *Left* in p_i , with all other nodes being in *Right*. If p_i does not start with a bidirected edge, p_i is a directed path, and p_i is also in *Right*. Since z_i is in p_j , and p_i and p_j have no sided intersection, the path must start with a bidirected edge (otherwise z_i is in both *Left* and *Right* - and thus cannot have an intersection). Similarly, p_j must start with a bidirected edge.

Furthermore, since p_i starts with a bidirected edge, the intersection with p_j must happen on a directed path to y . The same constraints apply to p_j . The resulting structure is shown in figure 5. Note that z_i and z_j can be directly connected by a bidirected edge, and in this case, both paths can traverse this edge. This case does not change our analysis.

We will construct alternate triples for z_i and z_j which do not intersect with each other. In particular, we will switch the paths of the two instrumental variables. That is, the triples $(z_i, p_i, W_i), (z_j, p_j, W_j)$ will be changed to $(z_i, p_j[z_i \sim y], W'_i), (z_j, p_i[z_j \sim y], W'_j)$. To prove that such modified triples exist, and satisfy theorem 8, several things need to be proved:

1. The modified paths have no sided intersection with each other, nor with other variables in the resulting instrumental set,
2. There exist W'_i and W'_j non-descendants of y , such that $p_j[z_i \sim y]$ and $p_i[z_j \sim y]$ respectively are not blocked, and both $(z_i \perp\!\!\!\perp y | W'_i)_{G_{E-}}$ and $(z_j \perp\!\!\!\perp y | W'_j)_{G_{E-}}$.

Notice that if these conditions are satisfied, the resulting set satisfies theorem 8.

For notational simplicity, we define

$$p'_i \equiv p_j[z_i \sim y] \text{ and } p'_j \equiv p_i[z_j \sim y]$$

which gives us new triples: (z_i, p'_i, W'_i) and (z_j, p'_j, W'_j) .

We first show that there is no sided intersection. Note that p'_i and p'_j are sub-paths of the original p_j and p_i , which by assumption have no sided intersection with any other paths in the instrumental set. The only modification now is that the paths start at z_i and z_j respectively. No path intersects with z_i or z_j in the new triples, because originally z_i and z_j were the intersection of two paths, one in *Left* and one in *Right*, meaning that no other path could go through them - and now this intersection no longer exists, and all other variables are unchanged. Thus the modified paths have no sided intersection with any other variable.

Finally, we show that there exist conditioning sets that satisfy the second requirement. We focus on W'_i , and W'_j will hold by symmetry.

We divide into two possible cases: $W_j \cap Desc(z_i)_{G_{E-}} \neq \emptyset$ and $W_j \cap Desc(z_i)_{G_{E-}} = \emptyset$.

- $W_j \cap Desc(z_i)_{G_{E-}} \neq \emptyset$ - Note that W_j does not block p'_i , since it doesn't block p_j . Now, suppose for the sake of contradiction $(z_i \not\perp\!\!\!\perp y | W_j)_{G_{E-}}$. This means that z_i is not d-separated from y in G_{E-} , so there exists an unblocked path from y to z_i . But since W_j conditions on a descendant of z_i , no matter how the path gets to z_i , it

can cross a collider at z_i , and be extended by $p_j[z_j \sim z_i]$, meaning that $(z_j \not\perp\!\!\!\perp y | W_j)_{G_{E-}}$, a contradiction. Finally, W_j does not contain descendants of y . Therefore, we can use $W'_i = W_j$.

- $W_j \cap Desc(z_i)_{G_{E-}} = \emptyset$ - In this case, we know that $y \notin Desc(z_i)_{G_{E-}}$, because if it were, we could create a path from y to z_j through z_i , since W_j does not condition on descendants of z_i , and W_j does not block $p_j[z_j \sim z_i]$, meaning that $(z_j \not\perp\!\!\!\perp y | W_j)_{G_{E-}}$, a contradiction. Consider $W'_i = W_i \setminus Desc(z_i)$. W'_i does not block p'_i , since p'_i is a directed path to the descendants of z_i . Finally, we need to show that $(z_i \perp\!\!\!\perp y | W'_i)_{G_{E-}}$. Suppose not. This means that there exists a path p_v from y to z_i which is not blocked by W'_i . We know that this path is blocked by W_i , so the blocking variable v must be a descendant of z_i . Since the path starts at y , which is not a descendant of z_i and goes to a descendant of z_i , it must come into the descendants of z_i through an incoming edge. This path must now get to z_i , but W'_i has no conditioning in the descendants of z_i , so p_v cannot cross a collider - but the graph is acyclic, so p_v cannot get to z_i by following a directed path in z_i 's descendants. But the path must get to z_i - a contradiction. Therefore, $(z_i \perp\!\!\!\perp y | W'_i)_{G_{E-}}$.

Since the conditions of theorem 8 are satisfied for the new set, we can perform this procedure for all pairs of variables which intersect. The procedure will only need to be done at most once per pair of variables, since the resulting paths cannot increase the number of double-intersections. The result is a set where $\forall z_i, z_j$, if z_j is on path p_i , then z_i is not on path p_j . \square

Theorem 7 requires a valid ordering of the variables. We showed that there are orderings of size 2, but in order to prove the theorem in general, we must show that there is a full ordering of all of the variables. To show this, we will first show that we can generate a set without intersection loops.

Definition 9. An intersection loop is a sequence of half-treks p_1, \dots, p_j where $\forall i$, p_i 's *Right* intersects with p_{i+1} 's *Left*, and p_j 's *Right* intersects with p_1 's *Left*.

An example of an intersection loop of size 3 is given in figure 6. Remember that the paths are half-treks WLOG, so intersection loops are the only type of loop possible. Thankfully, the next lemma shows that any instrumental set can be modified such that there is no intersection loop.

Lemma 14. There exist triples satisfying lemma 12 iff there exist triples satisfying the lemma, AND there are no intersection loops between $\{p_1, \dots, p_k\}$.

Proof. We will generalize the proof of lemma 13 to work with an arbitrary amount of nodes. Using the same arguments as given in lemma 13, the paths must all start with bidirected edges, and the only intersection allowed is between the first element of each path, and the directed portion of other paths.

Suppose there is an intersection loop of size n , consisting of p_1, \dots, p_n , with corresponding triples $(z_1, p_1, W_1), (z_2, p_2, W_2), \dots, (z_n, p_n, W_n)$. We claim

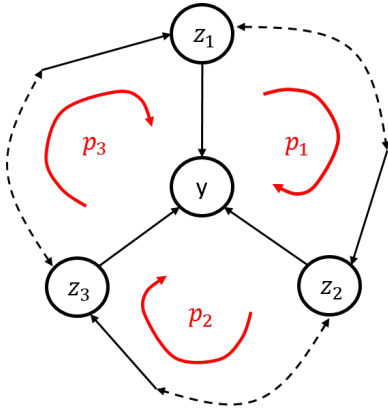


Figure 6: An example of an intersection loop of size 3. Note that in this case there is no ordering of all 3 variables $i < j$ s.t. z_j does not appear in path p_i .

that these triples can be replaced a new set: $(z_1, p_n[z_1 \sim y], W'_1), (z_2, p_1[z_2 \sim y], W'_2), \dots, (z_n, p_{n-1}[z_n \sim y], W'_n)$.

First note that each of the new paths is valid (since the original paths were half-treks, and intersected from *Right*, meaning that $p_i[z_{i+1} \sim y]$ is a directed path from z_{i+1} to y). These new paths have no sided intersection (see lemma 13). Furthermore, these new paths cannot be part of any intersection loop, since none of them start with bidirected edges. This means that we only need to do one pass through all the loops in the original instrumental set to remove them all.

Finally, we mirror the arguments given in the proof of lemma 13 to show that there exist new weights for each p_i that satisfy the conditions of lemma 12. Consider W'_i , for all $i = 1 \dots n$. We divide into two possible cases:

- $W_{i-1} \cap Desc(z_i)_{GE-} \neq \emptyset$ - Using the same argument as in lemma 13, $W'_i = W_{i-1}$ satisfies the requirements.
- $W_{i-1} \cap Desc(z_i)_{GE-} = \emptyset$ - Using the same argument as in lemma 13, $W'_i = W_i \setminus Desc(z_i)$ satisfies the requirements.

Since the new set satisfies the requirements of lemma 12, and the loop no longer exists, we can iteratively repeat the procedure for all intersection loops remaining in the instrumental set, taking apart at most $\frac{k}{2}$ loops (if all paths are part of a loop of size 2). We are then left with a graph with no intersection loops. □

Theorem 9. *There exists a set of triples satisfying theorem 7 if and only if there exists a set of triples satisfying theorem 8.*

Proof. \Leftarrow The first two conditions are identical. The only difference is the third condition. The indexing in this condition is irrelevant to this direction. Suppose that there is no intersection - then we have automatic satisfaction of lemma 12 and this theorem. If there is an intersection between two paths, then they share a variable V , and both $p_i[V \sim Y]$ and $p_j[Z_j \sim V]$ point to V . Since $p_i[V \sim Y]$ points to V , $V \in Left(p_i)$, and since the path is unblocked, it must point on to z_i , so $V \notin Right(p_i)$.

Similarly, $p_j[Z_j \sim V]$ points to V , meaning that $V \in Right(p_j)$, and the path is unblocked, so it must go from V to x_j , so $V \notin Left(p_j)$. Therefore the two paths have no sided intersection.

\Rightarrow The first two conditions are identical. We will focus on condition 3. Using lemma 12 and corollary 2, we can generate a set of triples which have no intersection except at the instrumental variables z . Since z_i is in *Left*, any intersection must be in *Right* of the intersecting path. This means that both $p_i[z_i \sim y]$ and $p_j[z_j \sim z_i]$ point to z_i , satisfying the second part of the third condition.

We generate an ordering for the variables by generating a directed intersection graph, where there is a directed arrow between p_i and p_j if z_j appears in path p_i . Note that z_j appears in path p_i iff p_j 's *Left* intersects with p_i 's *Right*. By lemma 14, this graph is acyclic. We therefore can put the nodes in topological order, giving us an ordering satisfying theorem 7. □

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