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Causal Effect Identification by Adjustment under Confounding and Selection Biases

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Abstract

Controlling for selection and confounding biases are two of the most challenging problems in the empirical sciences as well as in artificial intelligence tasks. Covariate adjustment (or, Backdoor Adjustment) is the most pervasive technique used for controlling confounding bias, but the same is oblivious to issues of sampling selection. In this paper, we introduce a generalized version of covariate adjustment that simultaneously controls for both confounding and selection biases. We first derive a sufficient and necessary condition for recovering causal effects using covariate adjustment from an observational distribution collected under preferential selection. We then relax this setting to consider cases when additional, unbiased measurements over a set of covariates are available for use (e.g., the age and gender distribution obtained from census data). Finally, we present a complete algorithm with polynomial delay to find all sets of admissible covariates for adjustment when confounding and selection biases are simultaneously present and unbiased data is available.

Introduction

One of the central challenges in data-driven fields is to compute the effect of interventions – for instance, how increasing the educational budget will affect violence rates in a city, whether treating patients with a certain drug will help their recovery, or how increasing the product price will change monthly sales? These questions are commonly referred as the problem of identification of causal effects. There are two types of *systematic bias* that pose obstacles to this kind of inference, namely *confounding bias* and *selection bias*. The former refers to the presence of a set of factors that affect both the action (also known as treatment) and the outcome (Pearl 1993), while the latter arises when the action, outcome, or other factors differentially affect the inclusion of subjects in the data sample (Bareinboim and Pearl 2016).

The goal of our analysis is to produce an unbiased estimand of the *causal effect*, specifically, the probability distribution of the outcome when an action is performed by an autonomous agent (e.g., FDA, robot), regardless of how the decision would naturally occur (Pearl 2000, Ch. 1). For example, consider the graph in Fig. 1(a) in which X represents a treatment (e.g., taking or not a drug), Y represents an outcome (health status), and Z is a factor (e.g., gender, age) that affects both the propensity of being treated and the outcome. The edges (Z, X) and (Z, Y) may encode the facts "gender affects how the drug is being prescribed" and "gender affects recovery" respectively – for example, females may be more health conscious, so they seek for treatment more frequently than their male counterparts and at the same time are less likely to develop large complications for the particular disease. Intuitively, the causal effect represents the variations of X that bring about change in Y *regardless* of the influence of Z on X, which is graphically represented in Fig. 1(b). Mutilation is the graphical operation of removing arrows representing a decision made by an autonomous agent of setting a variable to a certain value. The mathematical counterpart of mutilation is the $do()$ operator and the average causal effect of X on Y is usually written in terms of the *do*-distribution $P(y \mid do(x))$ (Pearl 2000, Ch. 1).

The gold standard for obtaining the *do*-distribution is through the use of randomization, where the treatment assignment is selected by a randomized device (e.g., a coin flip) regardless of any other set of covariates (Z). In fact, this operation physically transforms the reality of the underlying population (Fig. 1(a)) into the corresponding mutilated world (Fig. 1(b)). The effect of Z on X is neutralized once randomization is applied. Despite its effectiveness, randomized studies can be prohibitively expensive, and even unattainable in certain cases, either for technical, ethical, or technical reasons – e.g., one cannot randomize the cholesterol level of a patient and record if it causes the heart to stops, when trying to assess the effect of cholesterol level on cardiac failure.

An alternative way of computing causal effects is trying to relate non-experimentally collected samples (drawn from $P(z, x, y)$) with the experimental distribution ($P(y \mid do(x))$). Non-experimental (often called observational) data relates to the model in Fig. 1(a) where subjects decide by themselves to take or not the drug (X) while influenced by other factors (Z). There are a number of techniques developed for this task, where the most general one is known as *do-calculus* (Pearl 1995). In practice, one particular strategy from do-calculus called *adjustment* is used the most. It consists of averaging the effect of X on Y over the different levels of Z, isolating the effect of interest from the effect induced by other factors. Controlling for confounding bias by adjustment is currently the standard method for inferring causal effects in data-driven fields, and different properties and enhancements have been studied in statistics (Rubin 1974; Robinson and Jewell 1991; Pirinen, Donnelly, and Spencer 2012; Mefford and Witte 2012) and AI (Pearl 1993; 1995; Pearl and Paz 2010; Shpitser, VanderWeele, and Robins 2010; Maathuis and Colombo 2015; van der Zander, Liskiewicz, and Textor 2014).

Orthogonal to confounding, *sampling selection bias* is induced by preferential selection of units for the dataset, which is usually governed by unknown factors including treatment, outcome, and their consequences. It cannot be removed by a randomized trial and may stay undetected during the data gathering process, the whole study, or simply never be detected[1]. Consider Fig. 1(e) where X and Y represent again treatment and outcome, but S represents a binary variable that indicates if a subject is included in the pool (S=1 means that the unit is in the sample, S=0 otherwise). The effect of X on Y in the entire population ($P(y \mid do(x))$) is usually not the same as in the sample ($P(y \mid do(x), S{=}1)$). For instance, patients that went to the hospital and were sampled are perhaps more affluent and have better nutrition than the average person in the population, which can lead to a faster recovery. This preferential selection of samples challenges the validity of inferences in several tasks in AI (Cooper 1995; Cortes et al. 2008; Zadrozny 2004) and Statistics (Little and Rubin 1986; Kuroki and Cai 2006) as well as in the empirical sciences (Heckman 1979; Angrist 1997; Robins 2001).

The problem of selection bias can be addressed by removing the influence of the biased sampling mechanism on the outcome as if a random sample of the population was taken. For the graph in Fig. 1(d), for example, the distribution $P(y \mid do(x))$ is equal to $P(y \mid x, S{=}1)$ because there are not external factors that affect X and the selection mechanism S is independent of the outcome Y when the effect is estimated for the treatment X. There exists a complete non-parametric[2] solution for the problem of estimating statistical quantities from selection biased datasets (Bareinboim and Pearl 2012), and also sufficient and algorithmic conditions for recovering from selection in the context of causal inference (Bareinboim, Tian, and Pearl 2014; Bareinboim and Tian 2015).

Both confounding and selection biases carry extraneous "flow" of information between treatment and outcome, which is usually deemed "spurious correlation" since it does not correspond to the effect we want to compute on. Despite all the progress made in controlling these biases separately, we show that to estimate causal effects considering both problems requires a more refined analysis. First, note that the effect of X on Y can be estimated by blocking confounding and controlling for selection, respectively, in Figs. 1(a) and (d). On the other hand, confounding cannot be removed in Fig. 1(c) nor it can be recovered from selection bias in Fig. 1(e). Perhaps surprisingly, Fig. 1(f) presents a scenario where either confounding or selection can be addressed separately ($P(y|do(x)) = \sum_Z P(y|x,z)P(z)$ and $P(z,y|do(x)) = P(z,y|do(x), S{=}1)$), but not simultaneously (without external data). As this example suggests, there is an intricate connection between these two biases that disallow the methods developed for these problems of being applied independently and then combined.

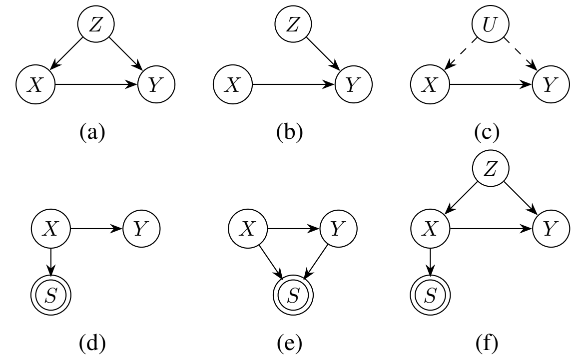


Figure 1: (a) and (d) give simple examples for confounding and selection bias respectively. (b) represents the model in (a) after an intervention is performed on X. (c) and (e) present examples where confounding and selection bias can not be removed respectively. In (f) we can control for either confounding or selection bias, but not for both unless we have external data on $P(z)$.

In this paper, we study the problem of estimating causal effects from models with an arbitrary structure that involve both biases. We establish necessary and sufficient conditions that a set of variables should fulfill so as to guarantee that the target effect can be unbiasedly estimated by adjustment. We consider two settings – first when only biased data is available, and then a more relaxed setting where additional unbiased samples of covariates are available for use (e.g., census data). Specifically, we solved the following problems:

1. **Identification and recoverability without external data**: The data is collected under selection bias, $P(\mathbf{v} \mid S{=}1)$, when does a set of covariates $\mathbf{Z}$ allow $P(\mathbf{y} \mid do(\mathbf{x}))$ to be estimated by adjusting for $\mathbf{Z}$?
2. **Identification and recoverability with external data**: The data is collected under selection bias $P(\mathbf{v} \mid S{=}1)$ and unbiased samples of $P(\mathbf{t}), \mathbf{T} \subseteq \mathbf{V}$, are available. When does a set of covariates $\mathbf{Z} \subseteq \mathbf{T}$ license the estimation of $P(\mathbf{y} \mid do(\mathbf{x}))$ by adjusting for $\mathbf{Z}$?
3. **Finding admissible adjustment sets with external data**: How can we list all admissible sets $\mathbf{Z}$ capable of identifying and recovering $P(\mathbf{y} \mid do(\mathbf{x}))$, for $\mathbf{Z} \subseteq \mathbf{T} \subseteq \mathbf{V}$?

Preliminaries

The systematic analysis of confounding and selection biases requires a formal language where the characterization of the underlying data-generating model can be encoded explicitly.

[1](Zhang 2008) noticed some interesting cases where detection is feasible in a class of non-chordal graphs.

[2]No assumptions about the about the functions that relates variables are made (i.e. linearity, monotonicity).

We use the language of Structural Causal Models (SCM) (Pearl 2000, pp. 204-207). Formally, a SCM M is a 4-tuple $\langle U, V, F, P(u)\rangle$, where U is a set of exogenous (latent) variables and V is a set of endogenous (measured) variables. F represents a collection of functions $F = \{f_i\}$ such that each endogenous variable $V_i \in V$ is determined by a function $f_i \in F$, where f_i is a mapping from the respective domain of $U_i \cup Pa_i$ to V_i, $U_i \subseteq U$, $Pa_i \subseteq V \backslash V_i$ (where Pa_i is the set of endogenous variables that are arguments of f_i), and the entire set F forms a mapping from U to V. The uncertainty is encoded through a probability distribution over the exogenous variables, $P(u)$. Within the structural semantics, performing an action $X{=}x$ is represented through the do-operator, $do(X{=}x)$, which encodes the operation of replacing the original equation of X by the constant x and induces a submodel M_x. For a detailed discussion on the properties of structural models, we refer readers to (Pearl 2000, Ch. 7).

We will represent sets of variables in bold. The causal effect of a set $\mathbf{X}$ when it is assigned a set of values $\mathbf{x}$, on a set $\mathbf{Y}$ when it is instantiated as $\mathbf{y}$ will be written as $P(\mathbf{y} \mid do(\mathbf{x}))$, which is a short hand notation for $P(\mathbf{Y}{=}\mathbf{y} \mid do(\mathbf{X}{=}\mathbf{x}))$. Mainly, we will operate with $P(\mathbf{v})$, $P(\mathbf{v} \mid do(\mathbf{x}))$, $P(\mathbf{v} \mid S{=}1)$, respectively, the observational, experimental, and selection-biased distributions.

Formally, the task of estimating a probabilistic quantity from a selection-biased distribution is known as *recovering* from selection bias (Bareinboim and Pearl 2012). It is not uncommon for observations of a subset of the variables over the entire population (unbiased data) to be available for use. Therefore, we will consider two subsets of $\mathbf{V}$, $\mathbf{M}, \mathbf{T} \subseteq \mathbf{V}$, where $\mathbf{M}$ contains the variables for which data was collected under selection bias, and $\mathbf{T}$ encompasses the variables observed in the overall population, without bias. The absence of unbiased data is equivalent to have $\mathbf{T} = \emptyset$.

Selection Bias with Adjustment

The main justification for the validity of adjustment for confounding comes under a graphical conditions called the "Backdoor criterion" (Pearl 1993; 2000), as shown below:

Definition 1 (Backdoor Criterion (Pearl 2000))**.** A set of variables $\mathbf{Z}$ satisfies the Backdoor Criterion relative to a pair of variables (X, Y) in a directed acyclic graph G if:

(i) No node in $\mathbf{Z}$ is a descendant of X.

(ii) $\mathbf{Z}$ blocks every path between X and Y that contains an arrow into X.

The heart of the criterion lies in cond. (ii), where the set $\mathbf{Z}$ is required to block all the backdoor paths between X and Y that generate confounding bias. Furthermore, cond. (i) forbids the inclusion of descendants of X in $\mathbf{Z}$, which intends to avoid opening new non-causal paths. For example, the empty set is admissible for adjustment in Fig. 1(e), but adding S would not be allowed since it is a descendant of X and opens the non-causal path $X \rightarrow S \leftarrow Y$. On the other hand, even though S does not open any non-causal path in Fig. 1(f), the criterion does not allow it to be used for adjustment.

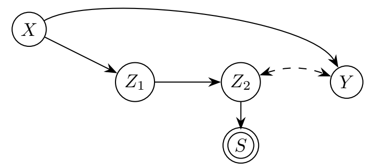


Figure 2: A graph that does not satisfy the s-backdoor criterion (respect to $\mathbf{Z}$), but the adjustment formula is recoverable and corresponds to desired causal effect.

(Bareinboim, Tian, and Pearl 2014) noticed that adjustment could be used for controlling for selection bias, in addition to confounding, which lead to a sufficient graphical condition called *Selection-Backdoor* criterion.

Definition 2 (Selection-Backdoor Criterion (Bareinboim and Tian 2015)). A set $\mathbf{Z} = \mathbf{Z}^+ \cup \mathbf{Z}^-$, with $\mathbf{Z}^- \subseteq De_X$ and $\mathbf{Z}^+ \subseteq \mathbf{V} \setminus De_X$ (where De_X is the set of variables that are descendants of X in G) satisfies the selection backdoor criterion (s-backdoor, for short) relative to X, Y and $\mathbf{M}, \mathbf{T}$ in a directed acyclic graph G if:

(i) $\mathbf{Z}^+$ blocks all back door paths from X to Y

(ii) X and $\mathbf{Z}^+$ block all paths between $\mathbf{Z}^-$ and Y, namely, $(\mathbf{Z}^- \perp\!\!\!\perp Y \mid X, \mathbf{Z}^+)$

(iii) X and $\mathbf{Z}$ block all paths between S and Y, namely, $(Y \perp\!\!\!\perp S \mid X, \mathbf{Z})$

(iv) $\mathbf{Z} \cup \{X, Y\} \subseteq \mathbf{M}$ and $\mathbf{Z} \subseteq \mathbf{T}$

The first two conditions echo the extended-backdoor (Pearl and Paz 2010)[3], while cond. (iii) and (iv) guarantee that the resultant expression is estimable from the available datasets. If the S-Backdoor criterion holds for $\mathbf{Z}$ relative to $\mathbf{X}, \mathbf{Y}$ and $\mathbf{M}, \mathbf{T}$ in G, then the effect $P(y \mid do(x))$ is identifiable, recoverable, and given by

$$P(y \mid do(x)) = \sum_{\mathbf{Z}} P(y \mid x, \mathbf{z}, S{=}1) P(\mathbf{z}) \tag{1}$$

We note here that the S-Backdoor is sufficient but not necessary for adjustment. To witness, consider the model in Fig. 2 where $\mathbf{Z} = \{Z_1, Z_2\}, \mathbf{M} = \{X, Y, Z_1, Z_2\}$, and $\mathbf{T} = \{Z_1, Z_2\}$. Here, $\mathbf{Z}^+ = \emptyset, \mathbf{Z}^- = \{Z_1, Z_2\}$. Condition (ii) in Def. 2 is violated, namely $(Z_1, Z_2 \not\!\perp\!\!\!\perp Y \mid X)$. Perhaps surprisingly, the effect $P(y \mid do(x))$ is identifiable and recoverable, as follows:

$$P(y|do(x)) = P(y|x) \tag{2}$$

$$= P(y|x) \sum\nolimits_{Z_1} P(z_1) \tag{3}$$

$$= \sum\nolimits_{Z_1} P(y|x, z_1) P(z_1) \tag{4}$$

$$= \sum\nolimits_{Z_1, Z_2} P(y|x, z_1, z_2) P(z_2|x, z_1) P(z_1) \tag{5}$$

$$= \sum\nolimits_{Z_1, Z_2} P(y|x, z_1, z_2) P(z_2|z_1) P(z_1) \tag{6}$$

$$= \sum\nolimits_{Z_1, Z_2} P(y|x, z_1, z_2, S{=}1) P(z_1, z_2) \tag{7}$$

[3]The extended-backdoor augments the backdoor criterion to allow for descendants of X that could be harmless in terms of bias.

(2) follows from the application of the second rule of do calculus and the independence $(X \perp\!\!\!\perp Y)_{G_{\underline{X}}}$. Equations (5),(6),(7) use the independences $(Y \perp\!\!\!\perp Z_1 | X)$, $(Z_2 \perp\!\!\!\perp X | Z_1)$ and $(S \perp\!\!\!\perp Y | X, Z_1, Z_2)$ respectively. The final expression (7) is estimable from the available data.

Considering that $\mathbf{Z} = \emptyset$ controls for confounding, adjusting for $\mathbf{Z} = \{Z_1, Z_2\}$ seems unnecessary. As it turns out, covariates irrelevant for confounding control, could play a role when we compound this task with recovering from selection bias (where Y will need to be separated from S).

Generalized Adjustment without External Data

Let us consider the case when only biased data $P(\mathbf{v} \mid S{=}1)$ over $\mathbf{V}$ is measured. Our interest in this section is on conditions that allow $P(\mathbf{y} \mid do(\mathbf{x}))$ to be computed by adjustment without external measurements.

Consider the model G in Fig. 3(a). Note that Y and S are marginally independent in $G_{\overline{X}}$ (the graph after an intervention on X where all edges into X are not present). As for confounding, Z needs to be conditioned on, but doing so opens a path between Y and S, letting spurious correlation from the bias to be included in our calculation. It turns out that with a careful manipulation of the expression, both biases can be controlled as follows:

$$P(y \mid do(x)) = P(y \mid do(x), S{=}1) \quad (8)$$

$$= \sum_Z P(y \mid do(x), z, S{=}1) P(z \mid do(x), S{=}1) \quad (9)$$

$$= \sum_Z P(y \mid x, z, S{=}1) P(z \mid S{=}1) \quad (10)$$

Eq. (8) follows from the independence $(Y \perp\!\!\!\perp S \mid X)$ in the mutilated graph $G_{\overline{X}}$. Next we condition on Z and the (10) is valid by the application of the second rule of do-calculus to the first term and the third rule to the second in (9). Note that every term in (10) is estimable from the biased distribution.

Next we introduce a complete criterion to determine whether adjusting by a set of covariates is admissible to identify and recover the causal effect. Before that, we require the concept of *proper causal path*.

Definition 3 (Proper Causal Path (Shpitser, VanderWeele, and Robins 2010))**.** Let $\mathbf{X}, \mathbf{Y}$ be sets of nodes. A causal path from a node in $\mathbf{X}$ to a node in $\mathbf{Y}$ is called proper if it does not intersect $\mathbf{X}$ except at the end point.

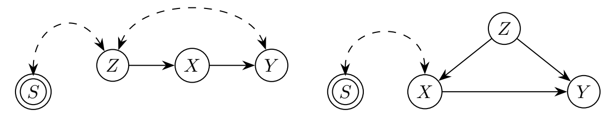
Definition 4 (Generalized Adjustment Criterion Type 1)**.** A set $\mathbf{Z}$ satisfies the generalized criterion relative to $\mathbf{X}, \mathbf{Y}$ in a causal model with graph G augmented with the selection mechanism S if:

(a) No element of $\mathbf{Z}$ is a descendant in $G_{\overline{X}}$ of any $W \notin \mathbf{X}$ which lies on a proper causal path from $\mathbf{X}$ to $\mathbf{Y}$.

(b) All non-causal paths between $\mathbf{X}$ and $\mathbf{Y}$ in G are blocked by $\mathbf{Z}$ and S .

(c) $\mathbf{Y}$ is d-separated from S given $\mathbf{X}$ under the intervention $do(\mathbf{x})$, i.e., $(\mathbf{Y} \perp\!\!\!\perp S \mid \mathbf{X})_{G_{\overline{\mathbf{X}}}}$.

(d) Every $X \in \mathbf{X}$ is either a non-ancestor of S or it is independent of $\mathbf{Y}$ in $G_{\underline{\mathbf{X}}}$, i.e., $\forall_{X \in \mathbf{X} \cap An_S} (X \perp\!\!\!\perp \mathbf{Y})_{G_{\underline{X}}}$



(a) (b)

Figure 3: Models where $\mathbf{Z}$ satisfies Def. 8

$G_{\overline{X(S)}}$ is the graph where all edges into $X \in \mathbf{X} \setminus An_S$ are removed, where $An_{\mathbf{V}}$ stands for the set of ancestors of a node or set $\mathbf{V}$ in G.

Theorem 1 (Generalized Adjustment Formula Type 1)**.** *Given disjoint sets of variables $\mathbf{X}, \mathbf{Y}$ and $\mathbf{Z}$ in a causal model with graph G. The effect $P(\mathbf{y} \mid do(\mathbf{x}))$ is given by*

$$P(\mathbf{y} \mid do(\mathbf{x})) = \sum_{\mathbf{Z}} P(\mathbf{y} \mid \mathbf{x}, \mathbf{z}, S{=}1) P(\mathbf{z} \mid S{=}1) \quad (11)$$

in every model inducing G if and only if they satisfy the generalized adjustment criterion type 1 (Def. 8).

The proof of Theorem 1 is presented in the supplemental material due the the length constraints for the paper. Conditions (a) and (b) echo the Extended Backdoor/Adjustment Criterion (Pearl and Paz 2010; Shpitser, VanderWeele, and Robins 2010) and guarantee that $\mathbf{Z}$ is admissible for adjustment in the unbiased distribution. Condition (c) requires the outcome $\mathbf{Y}$ to be independent of the selection mechanism S without observing any covariate $\mathbf{Z}$. Intuitively, condition (d) ensures that the distribution of some covariates under selection bias are viable to control for confounding.

The model in Fig. 3(b) also satisfies Def. 8, in this case the derivation can be performed by first applying the backdoor and then introducing S in both terms of the expression. In general, Def. 8 / Thm. 1 encapsulate all possible derivations that allow one to recover from both selection and confounding biases.

Generalized Adjustment With External Data

A natural question that arises is whether additional measurements in the population level over the covariates can help identifying and recovering the desired causal effect. The following criterion tries to relax the previous results by leveraging the unbiased data available.

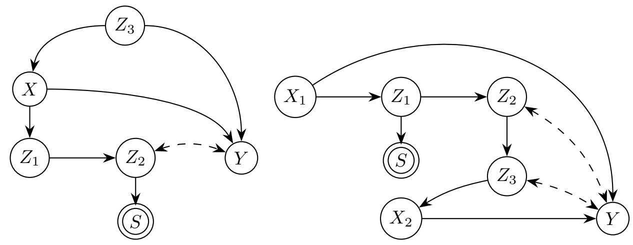
Definition 5 (Generalized Adjustment Criterion Type 2)**.** Let $\mathbf{T}$ be the set of variables measured without selection bias in the overall population. Also, let $\mathbf{X}, \mathbf{Y}, \mathbf{Z}$ be disjoint sets of variables in a causal model with diagram G, augmented with the selection mechanism S, where $\mathbf{Z} \subseteq \mathbf{T}$. Then, $\mathbf{Z}$ satisfies the generalized criterion relative to $\mathbf{X}, \mathbf{Y}$ if:

(a) No element in $\mathbf{Z}$ is a descendant in $G_{\overline{X}}$ of any $W \notin \mathbf{X}$ which lies on a proper causal path from $\mathbf{X}$ to $\mathbf{Y}$.

(b) All non-causal $\mathbf{X}$-$\mathbf{Y}$ paths in G are blocked by $\mathbf{Z}$.

(c) $\mathbf{Y}$ is independent of the selection mechanism S given $\mathbf{Z}$ and $\mathbf{X}$, i.e. $(\mathbf{Y} \perp\!\!\!\perp S \mid \mathbf{X}, \mathbf{Z})$

Theorem 2 (Generalized Adjustment Formula Type 2)**.** *Let $\mathbf{T}$ is the set of variables measured without selection bias.*



(a) (b)

Figure 4: Models where the set $\mathbf{Z}$ satisfies Def. 9.

Given disjoint sets of variables $\mathbf{X}, \mathbf{Y}$ and $\mathbf{Z} \subseteq \mathbf{T}$ and a causal diagram G, then, for every model inducing G, the effect $P(\mathbf{y} \mid do(\mathbf{x}))$ is given by

$$P(\mathbf{y} \mid do(\mathbf{x})) = \sum\nolimits_{\mathbf{Z}} P(\mathbf{y} \mid \mathbf{x}, \mathbf{z}, S{=}1) P(\mathbf{z}) \tag{12}$$

if and only if the set $\mathbf{Z}$ satisfies the generalized adjustment criterion type 2 relative to the pair $\mathbf{X}, \mathbf{Y}$.

Proof. Suppose the set $\mathbf{Z}$ satisfies the conditions relative to $\mathbf{X}, \mathbf{Y}$. Then, by conditions (a) and (b), for every model induced by G we have:

$$P(\mathbf{y} \mid do(\mathbf{x})) = \sum\nolimits_{\mathbf{Z}} P(\mathbf{y} \mid \mathbf{x}, \mathbf{z}) P(\mathbf{z})$$

We note that S can be introduced to the first term by cond. (c), which entail Eq. (34). The necessity part of the proof is more involved and is provided in the supplemental material (Correa and Bareinboim 2016). □

As in Def. 8, conditions (a) and (b) ensure $\mathbf{Z}$ is valid for adjustment without selection bias. Condition (c) requires that the influence of the selection mechanism in the outcome is nullified by conditioning on $\mathbf{X}$ and $\mathbf{Z}$ and condition (d) simply guarantees that the adjustment expression can be estimated from the available data. Fig. 4 presents two causal models that satisfies the previous criterion if measurements over $\mathbf{Z} = \{Z_1, Z_2, Z_3\}$ are available. To witness how the expression can be reached using do-calculus and probability axioms, consider Fig. 4(a):

$$P(y|do(x)) = \sum\nolimits_{Z_3} P(y|do(x), z_3) P(z_3|do(x)) \tag{13}$$

$$= \sum\nolimits_{Z_3} P(y \mid x, z_3) P(z_3) \tag{14}$$

$$= \sum\nolimits_{Z_1, Z_3} P(y \mid x, z_1, z_3) P(z_1, z_3) \tag{15}$$

$$= \sum\nolimits_{\mathbf{Z}} P(y \mid x, \mathbf{z}) P(z_2 \mid x, z_1, z_3) P(z_1, z_3) \tag{16}$$

$$= \sum\nolimits_{\mathbf{Z}} P(y \mid x, \mathbf{z}, S{=}1) P(\mathbf{z}) \tag{17}$$

We start by conditioning on Z_3 and removing $do(x)$ using rule 3 of the do-calculus. Then summing over Z_1 in the second term, pulling the new sum out, and introducing Z_1 into the first term, using $(Y \perp\!\!\!\perp Z_1 \mid Z_3, X)$, results in (15). Eq. (16) follows from conditioning the first term on Z_2, and finally by removing X in the second term using the independence $(Z_2 \perp\!\!\!\perp X | Z_1, Z_3)$, combining the last two distributions over the Z's and introducing the selection bias term using the independence $(Y \perp\!\!\!\perp S \mid X, \mathbf{Z})$ results in (17), which corresponds to (34).

Model in Fig. 4(b) also satisfies the type 2 criterion and illustrates how this can be applied to models where $\mathbf{X}$ and $\mathbf{Y}$ may be sets of variables.

Finding Admissible Sets for Generalized Adjustment

A natural extension to the problem is how to systematically list admissible sets for adjustment, using the criteria discussed in the previous sections. This is specially important in practice where factors such as feasibility, cost, and statistical power relate to the choosing of a covariate set.

In order to perform this kind of task efficiently, (van der Zander, Liskiewicz, and Textor 2014) introduced a transformation of the model called the *Proper Backdoor Graph* and formulate a criterion equivalent to the Adjustment Criterion:

Definition 6 (Proper Backdoor graph)**.** Let $G = (\mathbf{V}, \mathbf{E})$ be a DAG, and $\mathbf{X}, \mathbf{Y} \subseteq \mathbf{V}$ be pairwise disjoint subsets of variables. The proper backdoor graph, denoted as $G^{pbd}_{\mathbf{XY}}$, is obtained from G by removing the first edge of every proper causal path form $\mathbf{X}$ to $\mathbf{Y}$.

Definition 7 (Constructive Backdoor Criterion (CBD))**.** Let $G = (\mathbf{V}, \mathbf{E})$ be a DAG, and $\mathbf{X}, \mathbf{Y} \subseteq \mathbf{V}$ be pairwise disjoint subsets of variables. The set $\mathbf{Z}$ satisfies the Constructive Backdoor Criterion relative to $(\mathbf{X}, \mathbf{Y})$ in G if:

i) $\mathbf{Z} \subseteq \mathbf{V} \setminus Dpcp(\mathbf{X}, \mathbf{Y})$ and
ii) $\mathbf{Z}$ d-separates $\mathbf{X}$ and $\mathbf{Y}$ in the proper backdoor graph $G^{pbd}_{\mathbf{XY}}$.

Where $Dpcp(\mathbf{X}, \mathbf{Y}) = De((De_{\overline{\mathbf{X}}}(\mathbf{X}) \setminus \mathbf{X}) \cap An_{\underline{\mathbf{X}}}(Y))$

The set $Dpcp(\mathbf{X}, \mathbf{Y})$ is exactly the set of nodes forbidden by the first condition in both our generalized criteria, and $G^{pbd}_{\mathbf{XY}}$ only contain $\mathbf{X}, \mathbf{Y}$ paths that need to be blocked.

Lemma 3 (Constructive Backdoor $\implies$ Generalized Adjustment Type 2)**.** *Any set $\mathbf{Z}$ satisfying the CDB applied to $G^{pbd}_{(\mathbf{X} \cup S)\mathbf{Y}}$ and $Dpcp(\mathbf{X} \cup S, \mathbf{Y}) \cup (\mathbf{V} \setminus \mathbf{T})$ relative to $\mathbf{X}, \mathbf{Y}$ in G also satisfies the Generalized Adjustment Criterion type 2.*

Proof. By the equivalence between the CBD criterion and the adjustment criterion, we have that $Dpcp(\mathbf{X}, \mathbf{Y})$ is exactly the set of nodes forbidden by cond. (a) of the type 2 criterion, so

$$\begin{aligned} &Dpcp(\mathbf{X} \cup S, \mathbf{Y}) \\ &\quad = De((De_{\overline{\mathbf{X}, S}}(\mathbf{X} \cup \{S\}) \setminus (\mathbf{X} \cup S)) \cap An_{\underline{\mathbf{X}, S}}(Y)) \end{aligned} \tag{18}$$

Since S has no descendants, $De_{\overline{\mathbf{X}, S}}(\mathbf{X} \cup \{S\}) = De_{\overline{\mathbf{X}}}(\mathbf{X}) \cup S$ and $An_{\underline{\mathbf{X}, S}}(Y) = An_{\underline{\mathbf{X}}}(Y)$. As a consequence $Dpcp(\mathbf{X} \cup S, \mathbf{Y}) = Dpcp(\mathbf{X}, \mathbf{Y})$ implying cond. (a) of Def. 9.

$G^{pbd}_{(\mathbf{X} \cup S)\mathbf{Y}}$ has all non-causal paths from $\mathbf{X}$ to $\mathbf{Y}$ present in $G^{pbd}_{\mathbf{XY}}$, therefore, if $\mathbf{Z}$ block all non-causal paths in the former, it will do in the latter satisfying condition (b).

Every S – $\mathbf{Y}$ path may or may not contain $\mathbf{X}$. If not, $\mathbf{Z}$ should block it in $G^{pbd}_{(\mathbf{X} \cup S)\mathbf{Y}}$. In the latter case, the subpath from $\mathbf{X}$ to $\mathbf{Y}$ is either causal or non-causal. If it is causal $\mathbf{Z}$ will not block it, but the S-$\mathbf{Y}$ path will be blocked by $\mathbf{X}$. If the subpath is non-causal $\mathbf{Z}$ should block it, therefore, the larger path is blocked too. This argument implies condition (c). Since CBD holds for $Dpcp(\mathbf{X} \cup S, \mathbf{Y}) \cup (\mathbf{V} \setminus \mathbf{T})$ every element in $\mathbf{Z}$ must belong to $\mathbf{T}$ satisfying condition (d). □

Lemma 4 (Generalized Adjustment Type 2 $\implies$ Constructive Backdoor)**.** *Any set $\mathbf{Z}$ satisfying the Generalized Adjustment Criterion type 2 relative to $\mathbf{X}, \mathbf{Y}$ in G also satisfies the constructive backdoor criterion applied to $G^{pbd}_{(\mathbf{X} \cup S)\mathbf{Y}}$ and* $Dpcp(\mathbf{X} \cup S, \mathbf{Y}) \cup (\mathbf{V} \setminus \mathbf{T})$.

Proof. By lemma 3, $Dpcp(\mathbf{X} \cup S, \mathbf{Y}) = Dpcp(\mathbf{X}, \mathbf{Y})$, which combined with condition (d) implies condition (i) of the CBP.

By cond. (b) every non-causal path from $\mathbf{X}$ to $\mathbf{Y}$ is blocked by $\mathbf{Z}$ and all paths from S to $\mathbf{Y}$ (which are always non-causal when S is treated as an $\mathbf{X}$) are blocked by $\mathbf{Z}, \mathbf{X}$ by cond. (c). Those two facts together imply cond. (ii) of the CBD. □

Theorem 5 (Generalized Adjustment Type 2 ⇔ Constructive Backdoor)**.** *A set $\mathbf{Z}$ satisfies the Generalized Adjustment Criterion type 2 relative to $\mathbf{X}, \mathbf{Y}$ in G if and only if it satisfies the CBC applied to $G^{pbd}_{(\mathbf{X} \cup \mathbf{S})\mathbf{Y}}$ and $Dpcp(\mathbf{X} \cup S, \mathbf{Y}) \cup$* $(\mathbf{V} \setminus \mathbf{T})$.

Proof. It follows immediately from lemmas 3,4. □

Thm. 5 allows us to use the LISTSEP procedure (van der Zander, Liskiewicz, and Textor 2014) to list all the valid sets for the generalized adjustment type 2. The algorithm guarantees $O(n(n+m))$ polynomial delay, where n is the number of nodes and m is the number of edges in G (see (Takata 2010)). That means that the time needed to output the first solution or indicate failure, and the time between the output of consecutive solutions, is $O(n(n+m))$.

To provide the reader an intuition of how the algorithm works, consider the graph in Fig. 5(a) and its associated constructive backdoor graph in (b). W is a "forbidden node" in the sense that it cannot be used for adjustment and for this example is the only element in $Dpcp(\mathbf{X}, Y)$ assuming that unbiased measurement on the covariates Z_1, Z_2 and Z_3 are available (i.e. $\{Z_1, Z_2, Z_3\} \subseteq \mathbf{T}$). The algorithm LISTSEP will output every set of variables that d-separates $X \cup S$ from Y in the proper backdoor graph that does not contain any node in $Dpcp(\mathbf{X}, Y)$.

Conclusions

We provide necessary and sufficient conditions for identification and recoverability from selection bias of causal effects by adjustment, applicable for data-generating models with latent variables and arbitrary structure in nonparametric settings. Def. 8 and Thm. 1 provide a complete characterization of identification and recoverability by adjustment when no external information is available. Def. 9 and Thm. 3 provide a complete graphical condition for when external information on a set of covariates is available. Thm. 5 allowed us to list all sets that satisfies the last criterion in polynomial-delay time, effectively helping in the decision of what covariates need to be measured for recoverability. This is especially important when measuring a variable is associated with a particular cost or effort. Despite the fact that adjustment is neither complete nor the only method to identify causal effects, it is in fact the most used tool in the empirical sciences. The methods developed in this paper should help to formalize and alleviate the problem of sampling selection and confounding biases in a broad range of data-intensive applications.

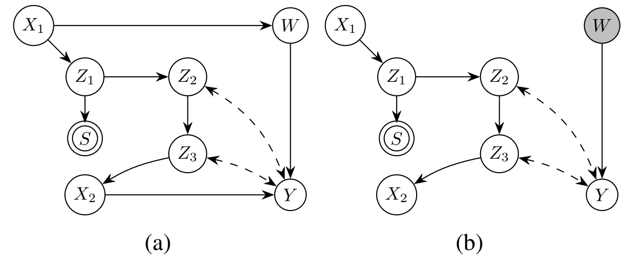


Figure 5: (a) shows a causal model and (b) the proper backdoor graph associated with it relative to $\mathbf{X} \cup S$ and Y. The gray nodes in (b) represents variables in *Dpcp*.

References

Angrist, J. D. 1997. Conditional independence in sample selection models. *Economics Letters* 54(2):103–112.

Bareinboim, E., and Pearl, J. 2012. Controlling selection bias in causal inference. In Lawrence, N., and Girolami, M., eds., *Proceedings of the 15th International Conference on Artificial Intelligence and Statistics (AISTATS)*. La Palma, Canary Islands: JMLR. 100–108.

Bareinboim, E., and Pearl, J. 2016. Causal inference and the data-fusion problem. *Proceedings of the National Academy of Sciences* 113:7345–7352.

Bareinboim, E., and Tian, J. 2015. Recovering causal effects from selection bias. *Proceedings of the Twenty-Ninth AAAI Conference on Artificial Intelligence* 3475–3481.

Bareinboim, E.; Tian, J.; and Pearl, J. 2014. Recovering from selection bias in causal and statistical inference. In Brodley, C. E., and Stone, P., eds., *Proceedings of the Twenty-eighth AAAI Conference on Artificial Intelligence*, 2410–2416. Palo Alto, CA: AAAI Press.

Cooper, G. 1995. Causal discovery from data in the presence of selection bias. In *Proceedings of the Fifth International Workshop on Artificial Intelligence and Statistics*, 140–150.

Correa, J. D., and Bareinboim, E. 2016. Causal effect identification by adjustment under confounding and selection biases - supplemental material. Technical report, Purdue AI Lab, Department of Computer Science, Purdue University.

Cortes, C.; Mohri, M.; Riley, M.; and Rostamizadeh, A. 2008. Sample selection bias correction theory. In *International Conference on Algorithmic Learning Theory*, 38–53. Springer.

Heckman, J. J. 1979. Sample selection bias as a specification error. *Econometrica* 47(1):153–161.

Kuroki, M., and Cai, Z. 2006. On recovering a population covariance matrix in the presence of selection bias. *Biometrika* 93(3):601–611.

Little, R. J. A., and Rubin, D. B. 1986. *Statistical Analysis with Missing Data*. New York, NY, USA: John Wiley & Sons, Inc.

Maathuis, M. H., and Colombo, D. 2015. A generalized back-door criterion. *Ann. Statist.* 43(3):1060–1088.

Mefford, J., and Witte, J. S. 2012. The covariate's dilemma. *PLoS Genet* 8(11):e1003096.

Pearl, J., and Paz, A. 2010. Confounding equivalence in causal equivalence. In *Proceedings of the Twenty-Sixth Conference on Uncertainty in Artificial Intelligence*. Corvallis, OR: AUAI. 433–441.

Pearl, J. 1993. Aspects of graphical models connected with causality. In *Proceedings of the 49th Session of the International Statistical Institute*, 391–401.

Pearl, J. 1995. Causal diagrams for empirical research. *Biometrika* 82(4):669–688.

Pearl, J. 2000. *Causality: Models, Reasoning, and Inference*. New York: Cambridge University Press. 2nd edition, 2009.

Pirinen, M.; Donnelly, P.; and Spencer, C. C. 2012. Including known covariates can reduce power to detect genetic effects in case-control studies. *Nature genetics* 44(8):848–851.

Robins, J. M. 2001. Data, design, and background knowledge in etiologic inference. *Epidemiology* 12(3):313–320.

Robinson, L. D., and Jewell, N. P. 1991. Some surprising results about covariate adjustment in logistic regression models. *International Statistical Review/Revue Internationale de Statistique* 227–240.

Rubin, D. 1974. Estimating causal effects of treatments in randomized and nonrandomized studies. *Journal of Educational Psychology* 66:688–701.

Shpitser, I.; VanderWeele, T. J.; and Robins, J. M. 2010. On the validity of covariate adjustment for estimating causal effects. In *Proceedings of UAI 2010*, 527–536.

Takata, K. 2010. Space-optimal, backtracking algorithms to list the minimal vertex separators of a graph. *Discrete Applied Mathematics* 158(15):1660–1667.

van der Zander, B.; Liskiewicz, M.; and Textor, J. 2014. Constructing separators and adjustment sets in ancestral graphs. In *Proceedings of UAI 2014*, 907–916.

Zadrozny, B. 2004. Learning and evaluating classifiers under sample selection bias. In *Proceedings of the twenty-first international conference on Machine learning*, 114. ACM.

Zhang, J. 2008. On the completeness of orientation rules for causal discovery in the presence of latent confounders and selection bias. *Artif. Intell.* 172:1873–1896.

Appendix

In order to prove the necessity of the criteria presented in the paper, it is imperative to construct structural causal models that serve as counter-examples to the identifiability or recoverability of the causal effect, whenever the set of covariates $\mathbf{Z}$ fails to satisfy the conditions relative to the pair $\mathbf{X}, \mathbf{Y}$. The following lemmata will be useful to construct such models. The first one, lemma 6 licenses the the direct specification of the conditional distributions of any variable given its parents, in accordance to the causal diagram G.

Lemma 6 (Family Parametrization)**.** *Let G be a causal diagram over a set $\mathbf{V}$ of n variables. Consider also, a set of conditional distributions $P(v_i \mid pa_{V_i}), 1 \leqslant i \leqslant n$ such that Pa_{V_i} is the set of nodes in G from which there are outgoing edges pointing into V_i. Then, there exists a model M compatible with G that induces $P(\mathbf{v}) = \prod_{i=1}^{n} P(v_i \mid pa_{V_i})$.*

Proof. (By construction) For every V_i define any ordering on the values of its domain, and let $v_i^{(j)}$ refer to the j^{th} value in that order. Also, define a continuous unobservable variable $U_i \sim U[0,1]$ (uniformly distributed in the interval $[0,1]$) for every variable $V_i \in \mathbf{V}$. Then, construct a structural causal model $M = \langle \mathbf{U}, \mathbf{V}, \mathcal{F}, P(\mathbf{u}) \rangle$ where:

- $\mathbf{V}$ is the same set of observables in G
- $\mathbf{U} = \bigcup_{i=1}^{n} U_i'$
- $\mathcal{F} = \left\{ f_i(pa_{V_i}, u_i) = \inf_j \left\{ \sum_{k=1}^{j} P(v_i^{(j)} \mid pa_{V_i}) \geqslant u_i \right\}, 1 \leqslant i \leqslant n \right\}$
- $U_i \sim U[0,1], 1 \leqslant i \leqslant n$

At every variable V_i, given a particular configuration of Pa_{V_i}, M simulates its value using the distribution $P(v_i \mid pa_{V_i})$. By the Markov property, the joint distribution will be equal to the product of those distributions. □

The following lemma, permits the construction of a structural causal model M compatible with a causal diagram G, using another model compatible with a related, but different, causal diagram G' where some arrows in a chain of variables have the reverse direction.

Lemma 7 (Chain Reversal)**.**

Proof. (By construction) Given M and any probability distribution $P(\mathbf{v})$ induced by it, compute the joint distribution $P(r_1, \ldots, r_\ell, t)$. Construct a new model M' with the same set of observable variables and identical functions for all variables but for $R_1, \ldots, R_\ell, T$. For those, assign the functions $f_{R_i}(r_{i-1}, U_{R_i}), 1 \leqslant i \leqslant \ell - 1$ as in lemma 6. Also, let $f_{R_\ell}(U_{R_\ell}) = U_{R_\ell}, P(U_{R_\ell}) = P(r_\ell)$. By lemma 6 the sub-models composed of $R_1, \ldots, R_\ell, T$ in M' and M produce the exact same distribution and since the set of parents and function for every other part of the model are exactly the same the overall distribution is identical. □

Finally, the following lemma allows to simplify the parametrization of an arbitrarily long chain of binary variables.

Lemma 8 (Collapsible Path Parametrization)**.** *Consider a causal diagram G and a probability distribution $P(\mathbf{v})$ induced by any SCM compatible with G. If G contains a chain $W_0 \to W_1 \to \ldots \to W_k$, where each W_i represents a binary random variable, for every $1 \leqslant i \leqslant k$ the only incoming edge to W_i is from W_{i-1}, and every conditional distribution $P(w_i \mid w_{i-1}) = p$, $P(w_i \mid \overline{w_{i-1}}) = q$, for some $0 < p, q < 1$. Then, the conditional distribution $P(w_k \mid w_0) = \frac{q-(p-1)(p-q)^k}{q-p+1}$, $P(w_k \mid \overline{w_0}) = \frac{q-q(p-q)^k}{q-p+1}$.*

Proof. Since $W_0, \ldots, W_k$ is a chain, the value of W_k is a function of W_0 when all other $W_1, \ldots, W_{k-1}$ are marginalized. All W_i, $1 \leqslant i \leqslant k$ are independent of any other variable given W_0. Therefore, the distribution $P(w_k \mid w_0)$ is equal to $\sum_{i=1}^{k-1} \prod_{i=1}^{k} P(w_i \mid w_{i-1})$, because any other variable can be removed from any product in this expression and summed out. This distribution can be calculated as the product of 2x2 matrices corresponding to the conditional distributions $P(w_i \mid w_{i-1})$ when encoded as $W_M = \begin{bmatrix} p & q \\ 1-p & 1-q \end{bmatrix}$. The product of k of such matrices is readily available if W_M is decomposed using its eigenvalues $\{1, p-q\}$ and eigenvectors $\left\{ \left[\frac{q}{(1-p)}, -1\right], [1,1] \right\}$:

$$P(w_k \mid w_0) = \sum_{i=1}^{k-1} \prod_{i=1}^{k} P(w_i \mid w_{i-1}) = (W_M)^k \tag{19}$$

$$= \begin{bmatrix} \frac{q-(p-1)(p-q)^k}{q-p+1} & \frac{q-q(p-q)^k}{q-p+1} \\ 1 - \frac{q-(p-1)(p-q)^k}{q-p+1} & 1 - \frac{q-q(p-q)^k}{q-p+1} \end{bmatrix} \tag{20}$$

□

Definition 8 (Generalized Adjustment Criterion Type 1)**.** A set $\mathbf{Z}$ satisfies the generalized criterion relative to $\mathbf{X}, \mathbf{Y}$ in a causal model with graph G augmented with the selection mechanism S if:

(a) No element of $\mathbf{Z}$ is a descendant in $G_{\overline{\mathbf{X}}}$ of any $W \notin \mathbf{X}$ which lies on a proper causal path from $\mathbf{X}$ to $\mathbf{Y}$.

(b) All non-causal paths between $\mathbf{X}$ and $\mathbf{Y}$ in G are blocked by $\mathbf{Z}$ and S.

(c) $\mathbf{Y}$ is d-separated from S given $\mathbf{X}$ under the intervention $do(\mathbf{x})$, i.e., $(\mathbf{Y} \perp\!\!\!\perp S \mid \mathbf{X})_{G_{\overline{\mathbf{X}}}}$.

(d) Every $X \in \mathbf{X}$ is either a non-ancestor of S or it is independent of $\mathbf{Y}$ in $G_{\underline{\mathbf{X}}}$, i.e., $\forall_{X \in \mathbf{X} \cap An_S} (X \perp\!\!\!\perp \mathbf{Y})_{G_{\underline{X}}}$

$G_{\overline{X(S)}}$ is the graph where all edges into $X \in \mathbf{X} \setminus An_S$ are removed, where $An_{\mathbf{V}}$ stands for the set of ancestors of a node or set $\mathbf{V}$ in G.

Theorem 1 (Generalized Adjustment Formula Type 1)**.** *Given disjoint sets of variables $\mathbf{X}, \mathbf{Y}$ and $\mathbf{Z}$ in a causal model with graph G. The effect $P(\mathbf{y} \mid do(\mathbf{x}))$ is given by*

$$P(\mathbf{y} \mid do(\mathbf{x})) = \sum\nolimits_{\mathbf{Z}} P(\mathbf{y} \mid \mathbf{x}, \mathbf{z}, S{=}1) P(\mathbf{z} \mid S{=}1) \tag{21}$$

in every model inducing G if and only if $\mathbf{X}, \mathbf{Y}, \mathbf{Z}$ satisfy the generalized adjustment criterion type 1 (Def. 8).

Lemma 2. *Let $\mathbf{X}, \mathbf{Y}, \mathbf{Z}$ be three disjoint sets of variables in the graph G. If $\mathbf{Z}$ satisfy the the conditions of the criterion type 1 relative to the pair $\mathbf{X}, \mathbf{Y}$, $\mathbf{Z}$ can be partitioned into sets of variables as follows:*

- $\mathbf{Z}_{\mathbf{nd}}^{\mathbf{Y},\mathbf{1}} = \{Z \mid Z \in \mathbf{Z} \setminus De_{\mathbf{X}}$ *and* $(Z \perp\!\!\!\perp \mathbf{Y} \mid \mathbf{X}, S)_{G_{\overline{\mathbf{X}}}}\}$
- $\mathbf{Z}_{\mathbf{nd}}^{\mathbf{X},\mathbf{1}} = \{Z \mid Z \in \mathbf{Z} \setminus De_{\mathbf{X}} \setminus \mathbf{Z}_{\mathbf{nd}}^{\mathbf{Y},\mathbf{1}}$ *and* $(Z \perp\!\!\!\perp \mathbf{X} \mid \mathbf{Z}_{\mathbf{nd}}^{\mathbf{Y},\mathbf{1}}, S)_{G_{\overline{\mathbf{X}(S)}}}\}$
- $\mathbf{Z}_{\mathbf{d}}^{\mathbf{Y}} = \{Z \mid Z \in \mathbf{Z} \cap De_{\mathbf{X}}$ *and* $(Z \perp\!\!\!\perp \mathbf{Y} \mid \mathbf{X}, \mathbf{Z}_{\mathbf{nd}}^{\mathbf{Y},\mathbf{1}}, \mathbf{Z}_{\mathbf{nd}}^{\mathbf{X},\mathbf{1}}, S)_{G_{\overline{\mathbf{X}}}}\}$
- $\mathbf{Z}_{\mathbf{d}}^{\mathbf{X}} = \mathbf{Z} \cap De_{\mathbf{X}} \setminus \mathbf{Z}_{\mathbf{d}}^{\mathbf{Y}}$
- $\mathbf{Z}_{\mathbf{nd}}^{\mathbf{Y},\mathbf{2}} = \{Z \mid Z \in \mathbf{Z} \setminus De_{\mathbf{X}} \setminus \mathbf{Z}_{\mathbf{nd}}^{\mathbf{Y},\mathbf{1}} \setminus \mathbf{Z}_{\mathbf{nd}}^{\mathbf{X},\mathbf{1}}$ *and* $(Z \perp\!\!\!\perp \mathbf{Y} \mid \mathbf{X}, \mathbf{Z}_{\mathbf{nd}}^{\mathbf{Y},\mathbf{1}}, \mathbf{Z}_{\mathbf{nd}}^{\mathbf{X},\mathbf{1}}, \mathbf{Z}_{\mathbf{d}}^{\mathbf{Y}}, \mathbf{Z}_{\mathbf{d}}^{\mathbf{X}}, S)_{G_{\overline{\mathbf{X}}}}\}$
- $\mathbf{Z}_{\mathbf{nd}}^{\mathbf{X},\mathbf{2}} = \mathbf{Z} \setminus De_{\mathbf{X}} \setminus \mathbf{Z}_{\mathbf{nd}}^{\mathbf{Y},\mathbf{1}} \setminus \mathbf{Z}_{\mathbf{nd}}^{\mathbf{X},\mathbf{1}}$

Where $(\mathbf{Z}_{\mathbf{d}}^{\mathbf{X}} \perp\!\!\!\perp \mathbf{X} \mid \mathbf{Z}_{\mathbf{d}}^{\mathbf{Y}}, \mathbf{Z}_{\mathbf{nd}}^{\mathbf{Y},\mathbf{1}}, \mathbf{Z}_{\mathbf{nd}}^{\mathbf{X},\mathbf{1}}, S)_{G_{\overline{\mathbf{X}(\mathbf{Z}_{\mathbf{d}}^{\mathbf{Y}}, S)}}}$ *and* $(\mathbf{Z}_{\mathbf{nd}}^{\mathbf{X},\mathbf{2}} \perp\!\!\!\perp \mathbf{X} \mid \mathbf{Z} \setminus \mathbf{Z}_{\mathbf{nd}}^{\mathbf{X},\mathbf{2}}, S)_{G_{\overline{\mathbf{X}(\mathbf{Z}_{\mathbf{d}}^{\mathbf{Y}}, \mathbf{Z}_{\mathbf{d}}^{\mathbf{X}}, S)}}}$ *hold.*

Proof. First, to show $(\mathbf{Z}_{\mathbf{nd}}^{\mathbf{X},\mathbf{2}} \perp\!\!\!\perp \mathbf{X} \mid \mathbf{Z} \setminus \mathbf{Z}_{\mathbf{nd}}^{\mathbf{X},\mathbf{2}}, S)_{G_{\overline{\mathbf{X}(\mathbf{Z}_{\mathbf{d}}^{\mathbf{Y}}, \mathbf{Z}_{\mathbf{d}}^{\mathbf{X}}, S)}}}$, we will assume, for the sake of contradiction that it is not the case. Then, there exists a covariate $Z' \in \mathbf{Z} \setminus De_{\mathbf{X}} \setminus \mathbf{Z}_{\mathbf{nd}}^{\mathbf{Y},\mathbf{1}} \setminus \mathbf{Z}_{\mathbf{nd}}^{\mathbf{X},\mathbf{1}} \setminus \mathbf{Z}_{\mathbf{nd}}^{\mathbf{Y},\mathbf{2}}$ that is d-connected to some $X \in \mathbf{X}$ by a path q. For Z' to not be in $\mathbf{Z}_{\mathbf{nd}}^{\mathbf{Y},\mathbf{1}}$, it must be d-connected to some $Y \in \mathbf{Y}$ by a path p that does not contain any collider, except, possibly Z' itself. In particular it may not contain S because of cond. *(c)*. For Z' not to be in $\mathbf{Z}_{\mathbf{nd}}^{\mathbf{Y},\mathbf{2}}$, p does not contain any covariates in $\mathbf{Z}_{\mathbf{nd}}^{\mathbf{Y},\mathbf{1}}, \mathbf{Z}_{\mathbf{nd}}^{\mathbf{X},\mathbf{1}}, \mathbf{Z}_{\mathbf{d}}^{\mathbf{Y}}, \mathbf{Z}_{\mathbf{d}}^{\mathbf{X}}$ or $\mathbf{Z}_{\mathbf{nd}}^{\mathbf{Y},\mathbf{2}}$ because, as p does not have colliders, any of those would close it. If X is not an ancestor of S, q should have arrows going out from X. Since Z' is a non-descendant of X, q would contain a collider, but for Z' not to be in $\mathbf{Z}_{\mathbf{nd}}^{\mathbf{X},\mathbf{1}}$ such collider must be an ancestor of S, contradicting the assumption that X is not an ancestor of S. Hence, X must be an ancestor of S.

If the path q have arrows into X, the junction of the paths q and p witnesses a violation to cond. *(d)* of the criterion unless Z' is a collider, in which case the same path is non-causal, proper and open after all variables in $\mathbf{Z}$ are observed, contradicting cond. *(b)*. If q have arrows going out from X, and given that Z' is not a descendant of X, S must be a collider in q for Z' not to be in $\mathbf{Z}_{\mathbf{nd}}^{\mathbf{X},\mathbf{1}}$. But, for cond. *(c)* to hold, Z' must be a descendant of a collider in p itself, because p has no other colliders. Then, observing Z' will activate the path constituted by the concatenation of q and p, contradicting cond. *(b)*. As a consequence, such Z' cannot exists, proving the first part claim.

Second, consider $(\mathbf{Z}_{\mathbf{d}}^{\mathbf{X}} \perp\!\!\!\perp \mathbf{X} \mid \mathbf{Z}_{\mathbf{d}}^{\mathbf{Y}}, \mathbf{Z}_{\mathbf{nd}}^{\mathbf{Y},\mathbf{1}}, \mathbf{Z}_{\mathbf{nd}}^{\mathbf{X},\mathbf{1}}, S)_{G_{\overline{\mathbf{X}(\mathbf{Z}_{\mathbf{d}}^{\mathbf{Y}}, S)}}}$ and note that the empty set satisfies it. For the case when $\mathbf{Z} \cap De_{\mathbf{X}} \cap \mathbf{Z}_{\mathbf{d}}^{\mathbf{Y}} \neq \emptyset$, assume for the sake of contradiction that the independence does not hold. Then, there exists $Z' \in \mathbf{Z}$ and some $X \in \mathbf{X}$ such that a directed path q goes from X to Z', and is open when $\mathbf{X}$ and S are observed (note that q cannot contain any non-descendant of $\mathbf{X}$). Since $Z' \notin \mathbf{Z}_{\mathbf{d}}^{\mathbf{Y}}$ there exists also a path p from Z' to some $Y \in \mathbf{Y}$ that is open when $\mathbf{X}, \mathbf{Z}_{\mathbf{nd}}^{\mathbf{Y},\mathbf{1}}, \mathbf{Z}_{\mathbf{nd}}^{\mathbf{X},\mathbf{1}}, \mathbf{Z}_{\mathbf{d}}^{\mathbf{Y}}, S$ are observed, and does not contain any node in $\mathbf{X}$.

The path p must contain a collider (possibly Z'); otherwise, the junction of q and p is a proper causal path containing Z' which contradicts cond. *(a)* of the criterion. If Z' is the collider itself cond. *(b)* will be violated unless a variable in $\mathbf{Z}_{\mathbf{nd}}^{\mathbf{Y},\mathbf{2}}$ or $\mathbf{Z}_{\mathbf{nd}}^{\mathbf{X},\mathbf{2}}$ closes q. But, the former cannot do it because any variable in q would not satisfy the definition of $\mathbf{Z}_{\mathbf{nd}}^{\mathbf{Y},\mathbf{2}}$, and similarly, no node in q can be in $\mathbf{Z}_{\mathbf{nd}}^{\mathbf{X},\mathbf{2}}$. Therefore, Z' cannot be the collider in p. This means, that there is a variable $W \in \{S\} \cup \mathbf{Z}_{\mathbf{d}}^{\mathbf{Y}} \cup \mathbf{Z}_{\mathbf{nd}}^{\mathbf{Y},\mathbf{1}} \cup \mathbf{Z}_{\mathbf{nd}}^{\mathbf{X},\mathbf{1}}$ is the collider in p. However, S cannot be the one, because it would violate cond. *(c)*. Similarly, W does not satisfy the definition of $\mathbf{Z}_{\mathbf{d}}^{\mathbf{Y}}$, because it is not independent of Y. And, W cannot be in $\mathbf{Z}_{\mathbf{nd}}^{\mathbf{Y},\mathbf{1}}$ or $\mathbf{Z}_{\mathbf{nd}}^{\mathbf{X},\mathbf{1}}$ because it is a descendant of X. As a consequence, such active path p is not possible, meaning that $Z' \in \mathbf{Z}_{\mathbf{d}}^{\mathbf{Y}}$ or does not exists. □

Proof. (Of Theorem 1). (if) Suppose $\mathbf{Z}$ satisfy the the conditions of the theorem relative to the pair $\mathbf{X}, \mathbf{Y}$. Then $\mathbf{Z}$ can be partitioned into several sets as in Lemma. 2. The causal effect is derived as follows:

First, by condition (c), $(\mathbf{Y} \perp\!\!\!\perp S \mid X)_{G_{\overline{\mathbf{X}}}}$ and S can be introduced into the expression:

$$P(\mathbf{y} \mid do(\mathbf{x})) = P(\mathbf{y} \mid do(\mathbf{x}), S{=}1) \tag{22}$$

By definition, $\mathbf{Z}_{\mathbf{nd}}^{\mathbf{Y},\mathbf{1}}$ can be introduced in the factor along with a sum over that variable:

$$P(\mathbf{y} \mid do(\mathbf{x})) = \sum_{\mathbf{Z}_{\mathbf{nd}}^{\mathbf{Y},\mathbf{1}}} P(\mathbf{y} \mid do(\mathbf{x}), \mathbf{z}_{\mathbf{nd}}^{\mathbf{Y},\mathbf{1}}, S{=}1) P(\mathbf{z}_{\mathbf{nd}}^{\mathbf{Y},\mathbf{1}} \mid S{=}1) \tag{23}$$

Conditioning on $\mathbf{Z}_{\mathbf{nd}}^{\mathbf{X},\mathbf{1}}$, it becomes:

$$P(\mathbf{y} \mid do(\mathbf{x})) = \sum_{\mathbf{Z}_{\mathbf{nd}}^{\mathbf{Y},\mathbf{1}}, \mathbf{Z}_{\mathbf{nd}}^{\mathbf{X},\mathbf{1}}} \Big[P(\mathbf{y} \mid do(\mathbf{x}), \mathbf{z}_{\mathbf{nd}}^{\mathbf{Y},\mathbf{1}}, \mathbf{z}_{\mathbf{nd}}^{\mathbf{x},\mathbf{1}}, S{=}1) \tag{24}$$
$$P(\mathbf{z}_{\mathbf{nd}}^{\mathbf{x},\mathbf{1}} \mid do(\mathbf{x}), \mathbf{z}_{\mathbf{nd}}^{\mathbf{Y},\mathbf{1}}, S{=}1) P(\mathbf{z}_{\mathbf{nd}}^{\mathbf{Y},\mathbf{1}} \mid S{=}1) \Big]$$

By definition of $\mathbf{Z}_{\mathbf{nd}}^{\mathbf{X},\mathbf{1}}$ the independence $(\mathbf{Z}_{\mathbf{nd}}^{\mathbf{X},\mathbf{1}} \perp\!\!\!\perp \mathbf{X} \mid \mathbf{Z}_{\mathbf{nd}}^{\mathbf{Y},\mathbf{1}}, S)_{G_{\overline{\mathbf{X}(S)}}}$ holds (note that $G_{\overline{\mathbf{X}(S)}} = G_{\overline{\mathbf{X}(\mathbf{Z}_{\mathbf{nd}}^{\mathbf{Y},\mathbf{1}}, S)}}$ since $\mathbf{Z}_{\mathbf{nd}}^{\mathbf{Y},\mathbf{1}}$ is not a descendant of X), and rule 3 of the do-calculus allow the removal of $do(\mathbf{x})$ from the second term of the previous expression, which allow to join the second and third factor together using the chain rule

$$P(\mathbf{y} \mid do(\mathbf{x})) = \sum_{\mathbf{Z}_{\mathbf{nd}}^{\mathbf{Y},\mathbf{1}}, \mathbf{Z}_{\mathbf{nd}}^{\mathbf{X},\mathbf{1}}} \Big[P(\mathbf{y} \mid do(\mathbf{x}), \mathbf{z}_{\mathbf{nd}}^{\mathbf{Y},\mathbf{1}}, \mathbf{z}_{\mathbf{nd}}^{\mathbf{X},\mathbf{1}}, S{=}1) \tag{25}$$
$$P(\mathbf{z}_{\mathbf{nd}}^{\mathbf{Y},\mathbf{1}}, \mathbf{z}_{\mathbf{nd}}^{\mathbf{X},\mathbf{1}} \mid S{=}1) \Big]$$

The second factor can be summed over $\mathbf{Z}_{\mathbf{d}}^{\mathbf{Y}}$, then pull the sum out and introduce $\mathbf{Z}_{\mathbf{d}}^{\mathbf{Y}}$ in the first factor

$$P(\mathbf{y} \mid do(\mathbf{x})) = \sum_{\mathbf{Z}_{\mathbf{nd}}^{\mathbf{Y},\mathbf{1}}, \mathbf{Z}_{\mathbf{nd}}^{\mathbf{X},\mathbf{1}}, \mathbf{Z}_{\mathbf{d}}^{\mathbf{Y}}} \Big[P(\mathbf{y} \mid do(\mathbf{x}), \mathbf{z}_{\mathbf{nd}}^{\mathbf{Y},\mathbf{1}}, \mathbf{z}_{\mathbf{nd}}^{\mathbf{X},\mathbf{1}}, \mathbf{z}_{\mathbf{d}}^{\mathbf{Y}}, S{=}1) \tag{26}$$
$$P(\mathbf{z}_{\mathbf{nd}}^{\mathbf{Y},\mathbf{1}}, \mathbf{z}_{\mathbf{nd}}^{\mathbf{X},\mathbf{1}}, \mathbf{z}_{\mathbf{d}}^{\mathbf{Y}} \mid S{=}1) \Big]$$

Conditioning the first factor on $\mathbf{Z}_{\mathbf{d}}^{\mathbf{X}}$, yields:

$$P(\mathbf{y} \mid do(\mathbf{x})) = \sum_{\mathbf{Z}_{\mathbf{nd}}^{\mathbf{Y},\mathbf{1}}, \mathbf{Z}_{\mathbf{nd}}^{\mathbf{X},\mathbf{1}}, \mathbf{Z}_{\mathbf{d}}^{\mathbf{Y}}, \mathbf{Z}_{\mathbf{d}}^{\mathbf{X}}} \Big[P(\mathbf{y} \mid do(\mathbf{x}), \mathbf{z}_{\mathbf{nd}}^{\mathbf{Y},\mathbf{1}}, \mathbf{z}_{\mathbf{nd}}^{\mathbf{X},\mathbf{1}}, \mathbf{z}_{\mathbf{d}}^{\mathbf{Y}}, \mathbf{z}_{\mathbf{d}}^{\mathbf{X}}, S{=}1)$$
$$P(\mathbf{z}_{\mathbf{d}}^{\mathbf{X}} \mid do(\mathbf{x}), \mathbf{z}_{\mathbf{nd}}^{\mathbf{Y},\mathbf{1}}, \mathbf{z}_{\mathbf{nd}}^{\mathbf{X},\mathbf{1}}, \mathbf{z}_{\mathbf{d}}^{\mathbf{Y}}, S{=}1)$$
$$P(\mathbf{z}_{\mathbf{nd}}^{\mathbf{Y},\mathbf{1}}, \mathbf{z}_{\mathbf{nd}}^{\mathbf{X},\mathbf{1}}, \mathbf{z}_{\mathbf{d}}^{\mathbf{Y}} \mid S{=}1) \Big] \tag{27}$$

By lemma 2, the independence $(\mathbf{Z}_{\mathbf{d}}^{\mathbf{X}} \perp\!\!\!\perp \mathbf{X} \mid \mathbf{Z}_{\mathbf{d}}^{\mathbf{Y}}, \mathbf{Z}_{\mathbf{nd}}^{\mathbf{Y},\mathbf{1}}, \mathbf{Z}_{\mathbf{nd}}^{\mathbf{X},\mathbf{1}}, S)_{G_{\overline{\mathbf{X}(\mathbf{Z}_{\mathbf{d}}^{\mathbf{Y}}, S)}}}$ holds and using rule 3 of do-calculus, the $\mathbf{X}$ operator can be removed from the second factor, which allows to join the same and the third factor together:

$$P(\mathbf{y} \mid do(\mathbf{x})) = \sum_{\mathbf{Z}_{\mathbf{nd}}^{\mathbf{Y},\mathbf{1}}, \mathbf{Z}_{\mathbf{nd}}^{\mathbf{X},\mathbf{1}}, \mathbf{Z}_{\mathbf{d}}^{\mathbf{Y}}, \mathbf{Z}_{\mathbf{d}}^{\mathbf{X}}} \Big[P(\mathbf{y} \mid do(\mathbf{x}), \mathbf{z}_{\mathbf{nd}}^{\mathbf{Y},\mathbf{1}}, \mathbf{z}_{\mathbf{nd}}^{\mathbf{X},\mathbf{1}}, \mathbf{z}_{\mathbf{d}}^{\mathbf{Y}}, \mathbf{z}_{\mathbf{d}}^{\mathbf{X}}, S{=}1)$$
$$P(\mathbf{z}_{\mathbf{nd}}^{\mathbf{Y},\mathbf{1}}, \mathbf{z}_{\mathbf{nd}}^{\mathbf{X},\mathbf{1}}, \mathbf{z}_{\mathbf{d}}^{\mathbf{Y}}, \mathbf{z}_{\mathbf{d}}^{\mathbf{X}} \mid S{=}1) \Big] \tag{28}$$

Summing over $\mathbf{Z}_{\mathbf{nd}}^{\mathbf{Y},\mathbf{2}}$ in the second factor, pulling the summation out, and introducing the same term into the first factor using the independence $(\mathbf{Z}_{\mathbf{nd}}^{\mathbf{Y},\mathbf{2}} \perp\!\!\!\perp \mathbf{Y} \mid \mathbf{X}, \mathbf{Z}_{\mathbf{nd}}^{\mathbf{Y},\mathbf{1}}, \mathbf{Z}_{\mathbf{nd}}^{\mathbf{X},\mathbf{1}}, \mathbf{Z}_{\mathbf{d}}^{\mathbf{Y}}, \mathbf{Z}_{\mathbf{d}}^{\mathbf{X}}, S)_{G_{\overline{\mathbf{X}}}}$, yields:

$$P(\mathbf{y} \mid do(\mathbf{x})) = \sum_{\mathbf{Z}_{\mathbf{nd}}^{\mathbf{Y},\mathbf{1}}, \mathbf{Z}_{\mathbf{nd}}^{\mathbf{X},\mathbf{1}}, \mathbf{Z}_{\mathbf{d}}^{\mathbf{Y}}, \mathbf{Z}_{\mathbf{d}}^{\mathbf{X}}, \mathbf{Z}_{\mathbf{nd}}^{\mathbf{Y},\mathbf{2}}} \Big[$$
$$P(\mathbf{y} \mid do(\mathbf{x}), \mathbf{z}_{\mathbf{nd}}^{\mathbf{Y},\mathbf{1}}, \mathbf{z}_{\mathbf{nd}}^{\mathbf{X},\mathbf{1}}, \mathbf{z}_{\mathbf{d}}^{\mathbf{Y}}, \mathbf{z}_{\mathbf{d}}^{\mathbf{X}}, \mathbf{Z}_{\mathbf{nd}}^{\mathbf{Y},\mathbf{2}}, S{=}1)$$
$$P(\mathbf{z}_{\mathbf{nd}}^{\mathbf{Y},\mathbf{1}}, \mathbf{z}_{\mathbf{nd}}^{\mathbf{X},\mathbf{1}}, \mathbf{z}_{\mathbf{d}}^{\mathbf{Y}}, \mathbf{z}_{\mathbf{d}}^{\mathbf{X}}, \mathbf{Z}_{\mathbf{d}}^{\mathbf{Y},\mathbf{2}} \mid S{=}1) \Big] \tag{29}$$

Conditioning on $\mathbf{Z}_{\mathbf{nd}}^{\mathbf{X},\mathbf{2}}$:

$$P(\mathbf{y} \mid do(\mathbf{x})) = \sum_{\mathbf{Z}} \Big[P(\mathbf{y} \mid do(\mathbf{x}), \mathbf{z}, S{=}1)$$
$$P(\mathbf{z}_{\mathbf{nd}}^{\mathbf{X},\mathbf{2}} \mid do(\mathbf{x}), \mathbf{z}_{\mathbf{nd}}^{\mathbf{Y},\mathbf{1}}, \mathbf{z}_{\mathbf{nd}}^{\mathbf{X},\mathbf{1}}, \mathbf{z}_{\mathbf{d}}^{\mathbf{Y}}, \mathbf{z}_{\mathbf{d}}^{\mathbf{X}}, \mathbf{Z}_{\mathbf{d}}^{\mathbf{Y},\mathbf{2}}, S{=}1)$$
$$P(\mathbf{z}_{\mathbf{nd}}^{\mathbf{Y},\mathbf{1}}, \mathbf{z}_{\mathbf{nd}}^{\mathbf{X},\mathbf{1}}, \mathbf{z}_{\mathbf{d}}^{\mathbf{Y}}, \mathbf{z}_{\mathbf{d}}^{\mathbf{X}}, \mathbf{Z}_{\mathbf{d}}^{\mathbf{Y},\mathbf{2}} \mid S{=}1) \Big] \tag{30}$$

By lemma 2, the independence $(\mathbf{Z}_{\mathbf{nd}}^{\mathbf{X},\mathbf{2}} \perp\!\!\!\perp \mathbf{X} \mid \mathbf{Z} \setminus \mathbf{Z}_{\mathbf{nd}}^{\mathbf{X},\mathbf{2}}, S)_{G_{\overline{\mathbf{X}(\mathbf{Z}_{\mathbf{d}}^{\mathbf{Y}},\mathbf{Z}_{\mathbf{d}}^{\mathbf{X}},S)}}}$ holds and using rule 3 of do-calculus, the $\mathbf{X}$ operator can be removed from the second factor, which allows to join the same and the third factor together:

$$P(\mathbf{y} \mid do(\mathbf{x})) = \sum_{\mathbf{Z}} P(\mathbf{y} \mid do(\mathbf{x}), \mathbf{z}, S{=}1) P(\mathbf{z} \mid S{=}1) \tag{31}$$

Conditions (a) and (b) imply $(\mathbf{Y} \perp\!\!\!\perp \mathbf{X} \mid \mathbf{Z}, S{=}1)_{G_{\underline{\mathbf{X}}}}$ and can be used together with rule 2 of do-calculus to remove of the $do()$ operator from the first factor, which results in the adjustment formula:

$$P(\mathbf{y} \mid do(\mathbf{x})) = \sum_{\mathbf{Z}} P(\mathbf{y} \mid \mathbf{x}, \mathbf{z}, S{=}1) P(\mathbf{z} \mid S{=}1) \tag{32}$$

(Only if) Note that condition (b) extends the second condition from the adjustment criterion but also requiring all non-causal paths to be blocked even when S is observed. Suppose conditions (a) and (b) do not hold, then for any model compatible with $G_{\overline{S}}$, which is also compatible with G, the adjustment formula is equal to $\sum_{\mathbf{Z}} P(\mathbf{y} \mid \mathbf{x}, \mathbf{z}) P(\mathbf{z})$. But by the adjustment criterion (Shpitser, VanderWeele, and Robins 2010) this expression will not be equal to $P(\mathbf{y} \mid do(\mathbf{x}))$.

To show the necessity for the extension of condition (b), and conditions (c) and (d), we construct counter examples for the identifiability and recoverability of the causal effect. In every case, let $\mathbf{V}$ represent all variables in the graph except for the selection mechanism S, and Q refer to the adjustment formula as in Eq. (32). We construct two SCMs M_1 and M_2, that induce probability distributions P_1 and P_2, respectively. M_1 and M_2 will be compatible with G, and agree in the probability distribution under selection bias

$$P_1(\mathbf{v} \mid S{=}1) = P_2(\mathbf{v} \mid S{=}1) \tag{33}$$

but Q_1 in M_1 will be different distribution than Q_2 in M_2. Let M_1 be compatible with G and M_2 with $G_{\overline{S}}$, making S independent from Pa_S in M_2 (i.e. $(\mathbf{V} \perp\!\!\!\perp S)_{P_2}$). Recoverability should hold for any parametrization, hence without loss of generality, all variables are assumed to be binary. The construction parametrizes P_1 through its factors (as in lemma 6) and then parametrizes P_2 to enforce (33). As a consequence, $P_2(\mathbf{v}) = P_2(\mathbf{v} \mid S{=}1)$.

Without loss of generality, our attention can be directed into the particular $Y' \in \mathbf{Y}$ not satisfying the condition, and on the causal effect for Y'. To do this, the constructed model will have every variable in $\mathbf{Y} \setminus \{Y'\}$ disconnected from the

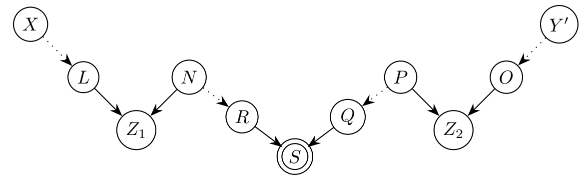


Figure 6: Non-causal path between $\mathbf{X}$ and $\mathbf{Y}$ activated when S and $\mathbf{Z}$ is observed. Edges with dotted lines encode arbitrarily long chains of nodes.

graph, more precisely $(\mathbf{Y} \setminus \{Y'\} \perp\!\!\!\perp \mathbf{V})$ holds, so that:

$$\begin{aligned}
P(\mathbf{y} \mid do(\mathbf{x})) &= \sum_{\mathbf{Z}} P(\mathbf{y} \mid \mathbf{x}, \mathbf{z}) P(\mathbf{z}) \\
&= \prod_{\mathbf{Y}} \sum_{\mathbf{Z}} P(y \mid \mathbf{x}, \mathbf{z}) P(\mathbf{z}) \\
&= \left(\prod_{\mathbf{Y} \setminus Y'} P(y) \right) \sum_{\mathbf{Z}} P(y' \mid \mathbf{x}, \mathbf{z}) P(\mathbf{z}) \\
&= \gamma \sum_{\mathbf{Z}} P(y' \mid \mathbf{x}, \mathbf{z}) P(\mathbf{z})
\end{aligned}$$

where γ represents the product of the marginal distribution of the remaining $\mathbf{Y} \setminus \{Y'\}$.

Suppose that condition (b) is not satisfied because there is some non-causal path from $X \in \mathbf{X}$ to $Y \in \mathbf{Y}$ blocked by $\mathbf{Z}$ but open when S is observed. The model in Fig. 6 contains a non-causal path that has arrows outgoing from X and Y. Models with arrow incoming to those variables belong to the same equivalence class as this one (lemma 7 can be use to reverse the directionality along chains of variables). It also contain one collider before and one after S. Models with more colliders in the non-causal path can be constructed in a similar way. Following the general construction described before, the value for the Q_1 and Q_2 are:

$$\begin{aligned}
Q_1 &= \gamma \sum_{\mathbf{Z}} P_1(y' \mid \mathbf{x}, \mathbf{z}) P_1(\mathbf{z}) = \\
&\gamma \sum_{\mathbf{Z}} P_1(y') P_1(\mathbf{z}) = \gamma P_1(y) \\
Q_2 &= \gamma \sum_{\mathbf{Z}} P_2(y' \mid \mathbf{x}, \mathbf{z}) P_2(\mathbf{z}) \\
&= \gamma \sum_{\mathbf{Z}} P_1(y' \mid \mathbf{x}, \mathbf{z}, S{=}1) P_1(\mathbf{z} \mid S{=}1) \\
&= \gamma \sum_{\mathbf{Z}} P_1(y' \mid \mathbf{x}, \mathbf{z}, S{=}1) P_1(\mathbf{z} \mid S{=}1) \\
&= \gamma \sum_{\mathbf{Z}} \frac{\sum_{L,N,R,Q,P,O} P_1(y', \mathbf{x}, \mathbf{z}, l, n, r, q, p, o, S{=}1)}{\sum_{Y',L,N,R,Q,P,O} P_1(y', \mathbf{x}, \mathbf{z}, l, n, r, q, p, o, S{=}1)} P_1(\mathbf{z} \mid S{=}1)
\end{aligned}$$

The numerator of the first term in the summation can be factorized

$$\begin{aligned}
&P_1(y', \mathbf{x}, \mathbf{z}, l, n, r, q, p, o, S{=}1) \\
&= P_1(\mathbf{x}) P_1(l \mid \mathbf{x}) P_1(z_1^* \mid l, n) P_1(n) P_1(r \mid n) P_1(S{=}1 \mid r, q) \\
&\quad P_1(q \mid p) P_1(z_2^* \mid p, o) P_1(p) P(o \mid y') P_1(y')
\end{aligned}$$

All the conditional distributions in the previous expression can be parametrized using lemma 6 and lemma 8 with parameters $p = 3/5, q = 2/5$, as follows: $P_1(\mathbf{x}) = P_1(y') = P_1(n) = P_1(p) = 1/2, P_1(z_1^* \mid l, n) = P_1(z_1^* \mid \bar{l}, \overline{n}) = 3/4, P_1(z_1^* \mid \bar{l}, n) = P_1(z_1^* \mid l, \overline{n}) = 1/4, P_1(z_2^* \mid \overline{p}, \overline{o}) = P_1(z_2^* \mid \overline{p}, o) = P_1(z_2^* \mid \overline{p}, o) = 1/2, P_1(z_2^* \mid p, o) = 3/4, P_1(S{=}1 \mid r, \overline{q}) = P_1(S{=}1 \mid \overline{r}, \overline{q}) = 1/2, P_1(S{=}1 \mid r, q) = 1/4, P_1(S{=}1 \mid \overline{r}, q) = 3/4, P_1(l \mid \mathbf{x}) = 1/2 + \epsilon_1/2, P_1(l \mid \overline{\mathbf{x}}) = 1/2 - \epsilon_1/2, P_1(r \mid n) = 1/2 + \epsilon_2/2, P_1(r \mid \overline{n}) = 1/2 - \epsilon_2/2, P_1(q \mid p) = 1/2 + \epsilon_3/2, P_1(q \mid \overline{p}) = 1/2 - \epsilon_3/2, P(o \mid y') = 1/2 + \epsilon_4/2, P(o \mid \overline{y'}) = 1/2 - \epsilon_4/2$, where $\epsilon_i = (1/5)^{k_i}$, and k_1 is the length of the path X to L, k_2 the length of the path from N to R, k_3 the length of the path from P to Q and k_4 the length of the path from Y' to O.

Calculating both Q_1 and Q_2 with this parametrization we obtain:

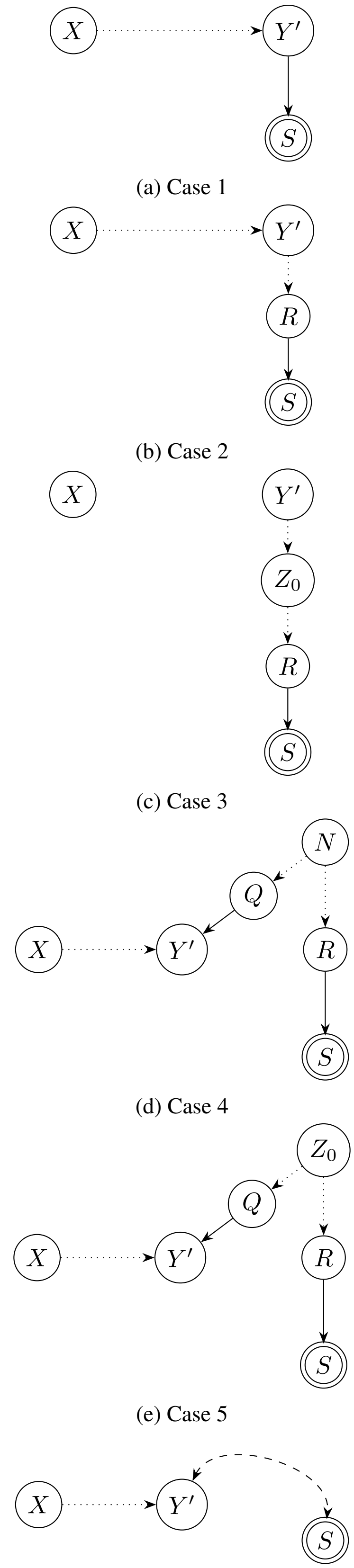
$$
\begin{aligned}
Q_1 &= \gamma/2 \\
Q_2 &= \gamma/2 \Bigg(\frac{14\epsilon_3\epsilon_4}{(\epsilon_1(7\epsilon_2 + \epsilon_2\epsilon_3) + 56)(\epsilon_3 + 7)} \\
&\quad - \frac{14\epsilon_3\epsilon_4}{(\epsilon_1(7\epsilon_2 + \epsilon_2\epsilon_3) - 56)(\epsilon_3 + 7)} \\
&\quad - \frac{2\epsilon_3 - \epsilon_3\epsilon_4 + \epsilon_3^2\epsilon_4 - \epsilon_3^2 + 63}{(\epsilon_3 + 7)(\epsilon_3 - 9)} \\
&\quad - \frac{18\epsilon_3\epsilon_4}{(\epsilon_1(9\epsilon_2 - \epsilon_2\epsilon_3) - 72)(\epsilon_3 - 9)} \\
&\quad + \frac{18\epsilon_3\epsilon_4}{(\epsilon_1(9\epsilon_2 - \epsilon_2\epsilon_3) + 72)(\epsilon_3 - 9)} \Bigg)
\end{aligned}
$$

Q_1 and Q_2 only when one of $\epsilon_i, i = 1, 2, 3, 4$ is equal to 0, which is never possible in this parametrization since every $\epsilon_i = (1/5)^{k_i}$ and every $k_i > 0$.

Suppose condition (c) does not hold, then, there is an open path between $\mathbf{Y}$ and S in $G_{\overline{X}}$. The following are the cases for which Y' may violate cond. (c). Figure 7 illustrates the structure of the cases graphically.

case 1: $Y' \in Pa_S$

Let $\mathbf{W}$ be the set of nodes connecting $\mathbf{X}$ and Y' with directed paths. Consider the induced subgraph G' where all nodes in $\mathbf{V} \setminus \{\mathbf{X}, \mathbf{W}, Y', S\}$ are disconnected from $\{\mathbf{X}, \mathbf{W}, Y', S\}$. It must be the case that $\mathbf{Z}$ and $\mathbf{W}$ are disjoint, else condition (a) is violated. Consequently, every $\mathbf{Z}$ is disconnected, and $(\mathbf{Z} \perp\!\!\!\perp Y')_{P_1}$ holds. M_1 and M_2 are constructed from G', the adjustment formula in the



(a) Case 1
(b) Case 2
(c) Case 3
(d) Case 4
(e) Case 5
(f) Case 6

Figure 7: Cases considered for the necessity of condition (c) in the proof for Thm. 1. Dotted directed arrows indicate chains of arbitrary length in the graph.

second model can be expressed as:

$$\begin{aligned}Q_2 &= \gamma \sum_{\mathbf{Z}} P_2(y' \mid \mathbf{x}, \mathbf{z}) P_2(\mathbf{z}) \\ &= \gamma \sum_{\mathbf{Z}} P_1(y' \mid \mathbf{x}, \mathbf{z}, S{=}1) P_1(\mathbf{z} \mid S{=}1) \\ &= \gamma \sum_{\mathbf{Z}} P_1(y' \mid \mathbf{x}, S{=}1) P_1(\mathbf{z} \mid S{=}1) \\ &= \gamma P_1(y' \mid \mathbf{x}, S{=}1) \\ &= \gamma \frac{P_1(y', \mathbf{x}, S{=}1)}{\sum_{Y'} P_1(y', \mathbf{x}, S{=}1)} \\ &= \gamma \frac{P_1(S{=}1 \mid y') P_1(y' \mid \mathbf{x})}{P_1(S{=}1 \mid y') P_1(y' \mid \mathbf{x}) + P_1(S{=}1 \mid \overline{y'}) P_1(\overline{y'} \mid \mathbf{x})}\end{aligned}$$

Using lemma 6, let $P_1(S{=}1 \mid y') = \alpha$ and $P_1(S{=}1 \mid \overline{y'}) = \beta$ with $0 < \alpha, \beta < 1$ and $\alpha \neq \beta$. Proceed with lemma 8 ($p = q = 1/2$) to define $P(y' \mid \mathbf{x}) = 1/2$. The previous expression becomes:

$$Q_2 = \gamma \frac{\alpha}{\alpha + \beta}$$

Following a similar derivation it can be established that $Q_1 = \gamma/2$ which is never equal to Q_2 in this parametrization.

case 2: There is a directed path p from Y' to S without any $\mathbf{Z}$.

Let R be the parent of S in such path, let $\mathbf{W_1}$ be the set of nodes between $\mathbf{X}$ and Y' as in the previous case. Similarly let $\mathbf{W_2}$ be the variables in the path from Y' to R. Now, consider the graph G' where all nodes except for $\{\mathbf{X}, \mathbf{W_1}, Y', \mathbf{W_2}, R, S\}$ are disconnected from $\{\mathbf{X}, \mathbf{W_1}, Y', \mathbf{W_2}, R, S\}$. Proceeding as in the previous case, with the consideration that $\mathbf{Z}$ is disconnected from the rest of the graph, yields:

$$Q_2 = \gamma P_1(y' \mid \mathbf{x}, S{=}1) = \gamma \frac{P_1(y', \mathbf{x}, S{=}1)}{P_1(\mathbf{x}, S{=}1)}$$

The numerator can be rewritten as:

$$\begin{aligned}P_1(y', \mathbf{x}, S{=}1) &= \sum_R P_1(y', \mathbf{x}, r, S{=}1) \\ &= \sum_R P_1(\mathbf{x}) P_1(y' \mid \mathbf{x}) P_1(r \mid y') P_1(S{=}1 \mid r)\end{aligned}$$

Factorizing the denominator analogously, the term $P_1(\mathbf{x})$ is the same and can be cancelled out, then Q_2 becomes:

$$Q_2 = \gamma \frac{P_1(y' \mid \mathbf{x}) \sum_R P_1(r \mid y') P_1(S{=}1 \mid r)}{\sum_{Y'} P_1(y' \mid \mathbf{x}) \sum_R P_1(r \mid y') P_1(S{=}1 \mid r)}$$

Using lemma 8 to set $P_1(r \mid y') = 1/2 + \epsilon/2, P_1(r \mid \overline{y'}) = 1/2 - \epsilon/2$ where $\epsilon = (1/5)^k$ (using $p = 3/5, q = 2/5$), and defining $P(S{=}1 \mid r) = 2/3$ and $P(S{=}1 \mid \overline{r}) = 1/2$ leads to $Q_2 = \gamma(1/2 + \epsilon/14)$ and $Q_1 = \gamma/2$ which are never equal.

case 3: There is a directed path p from Y' to S that contains some $Z_0 \in \mathbf{Z}$

Let R be the parent of S in such path. It can be assured that, $\mathbf{X}$ and Y' are not connected by any causal path, otherwise Z_0 violates condition (a). Let $\mathbf{W_1}$ be the nodes in the subpath between Y' and Z_0, and $\mathbf{W_2}$ those in between Z_0 and R. Consider the graph G' where all nodes except for $\{\mathbf{X}, Y', \mathbf{W_1}, Z_0, \mathbf{W_2}, R, S\}$ are disconnected from $\{\mathbf{X}, Y', \mathbf{W_1}, Z_0, \mathbf{W_2}, R, S\}$.
Every $\mathbf{Z}' = \mathbf{Z} \setminus \{Z_0\}$ is disconnected from the rest of the graph, then:

$$\begin{aligned}Q_2 &= \gamma \sum_{\mathbf{Z}} P_1(y' \mid \mathbf{x}, \mathbf{z}, S{=}1) P_1(\mathbf{z} \mid S{=}1) \\ &= \gamma \sum_{Z_0} \sum_{\mathbf{Z}'} P_1(y' \mid \mathbf{x}, z_0, \mathbf{z}', S{=}1) P_1(z_0, \mathbf{z}' \mid S{=}1) \\ &= \gamma \sum_{Z_0} P_1(y' \mid \mathbf{x}, z_0, S{=}1) P_1(z_0 \mid S{=}1) \\ &= \gamma \sum_{Z_0} P_1(y' \mid \mathbf{x}, z_0) P_1(z_0 \mid S{=}1) \\ &= \gamma \sum_{Z_0} \frac{P_1(y', \mathbf{x}, z_0)}{\sum_{Y'} P_1(y', \mathbf{x}, z_0)} P_1(z_0 \mid S{=}1)\end{aligned}$$

The numerator of the fraction in the last expression is equal to:

$$P_1(y', \mathbf{x}, z_0) = P_1(\mathbf{x}) P_1(y') P_1(z_0 \mid y')$$

A similar factorization can be employed for the denominator, as well as for Q_1. The factor $P_1(\mathbf{x})$ appears in both parts of the fractions and can be canceled:

$$\begin{aligned}Q_1 &= \gamma \sum_{Z_0} \frac{P_1(y') P_1(z_0 \mid y')}{\sum_{Y'} P_1(y') P_1(z_0 \mid y')} P_1(z_0) \\ Q_2 &= \gamma \sum_{Z_0} \frac{P_1(y') P_1(z_0 \mid y')}{\sum_{Y'} P_1(y') P_1(z_0 \mid y')} P_1(z_0 \mid S{=}1)\end{aligned}$$

Now, $P_1(z_0)$ and $P_1(z_0 \mid S{=}1)$ are derived in similar terms:

$$\begin{aligned}P_1(z_0) &= \sum_{Y'} P_1(z_0, y') = \sum_{Y'} P_1(y') P_1(z_0 \mid y') \\ P_1(z_0 \mid S{=}1) &= \frac{P_1(z_0, S = 1)}{P_1(S = 1)} \\ &= \frac{P_1(S = 1 \mid z_0) \sum_{Y'} P_1(y') P_1(z_0 \mid y')}{P_1(S = 1)}\end{aligned}$$

Replacing $P_1(z_0)$ and $P_1(z_0 \mid S{=}1)$ in Q_1 and Q_2, then simplifying:

$$\begin{aligned}Q_1 &= \gamma \sum_{Z_0} P_1(y') P(z_0 \mid y') = \gamma P_1(y') \\ Q_2 &= \gamma \frac{\sum_{Z_0} P_1(y') P_1(z_0 \mid y') P_1(S{=}1 \mid z_0)}{P_1(S = 1)} = \gamma \frac{P_1(y', S{=}1)}{P_1(S = 1)} \\ &= \gamma \frac{P_1(y') P_1(S{=}1 \mid y')}{\sum_{Y'} P_1(S{=}1 \mid y') P_1(y')}\end{aligned}$$

The term $P_1(S{=}1 \mid y') = \sum_R P_1(S{=}1 \mid r) P_1(r \mid y')$. Lemma 8 can be employed exactly as in the previous case, and $P_1(y')$ can be defined directly since it has no parents, for instance $P_1(y') = 1/2$, then the queries end up as:

$$Q_1 = \gamma \tfrac{1}{2} \qquad Q_2 = \gamma \left(\tfrac{1}{2} + \tfrac{\epsilon}{14}\right)$$

Which are never equal.

case 4: There is a path p connecting Y' and S that goes through and ancestor of both, and does not contain any node in $\mathbf{Z}$.

Let N be the closest common ancestor of Y' and S. Let R be the parent of S and Q the parent of Y' in the mentioned path. Let $\mathbf{W_1}$ be the set of nodes between $\mathbf{X}$ and Y'. Let $\mathbf{W_2}$ and $\mathbf{W_3}$ be the nodes in the paths from N to Q and from N to R respectively. Consider the graph G' where the arrows in the subpath from N to Q are reversed and all nodes except for $\{\mathbf{X}, \mathbf{W_1}, Y', Q, \mathbf{W_2}, N, \mathbf{W_3}, R, S\}$ are disconnected from $\{\mathbf{X}, \mathbf{W_1}, Y', Q, \mathbf{W_2}, N, \mathbf{W_3}, R, S\}$. Any model constructed for G' can be translated to a model compatible with G using lemma 7. Following the same derivation as in case 2 (taking into account that $\mathbf{Z}$ is disconnected from the rest of the graph) yields:

$$Q_2 = \gamma \frac{P_1(y', \mathbf{x}, S{=}1)}{\sum_{Y'} P_1(y', \mathbf{x}, S{=}1)}$$

The numerator of the last expression can be rewritten as:

$$\begin{aligned} P_1(y', \mathbf{x}, S{=}1) &= \sum_Q P_1(y', \mathbf{x}, q, S{=}1) \\ &= \sum_Q P_1(\mathbf{x}) P_1(y' \mid \mathbf{x}, q) P_1(q) P_1(S{=}1 \mid q) \end{aligned}$$

By rewriting the denominator similarly, the term $P_1(\mathbf{x})$ appearing in both vanishes, then Q_2 becomes:

$$\begin{aligned} Q_1 &= \gamma \frac{\sum_Q P_1(y' \mid \mathbf{x}, q) P_1(q)}{\sum_{Y',Q} P_1(y' \mid \mathbf{x}, q) P_1(q)} \\ Q_2 &= \gamma \frac{\sum_Q P_1(y' \mid \mathbf{x}, q) P_1(q) P_1(S{=}1 \mid q)}{\sum_{Y',Q} P_1(y' \mid \mathbf{x}, q) P_1(q) P_1(S{=}1 \mid q)} \end{aligned}$$

Lemma 8 can be employed to set $P_1(r \mid q) = 1/2 + \epsilon/2, P_1(r \mid \overline{q}) = 1/2 - \epsilon/2$ where $\epsilon = (1/5)^k$ (using $p = 3/5, q = 2/5$). Define $P(S{=}1 \mid r) = 2/3$ and $P(S{=}1 \mid \overline{r}) = 1/2$. Calculate $P(S{=}1 \mid q)$ as $\sum_R P_1(r \mid q) P_1(S{=}1 \mid r)$. Also by lemma 8 let $P_1(y' \mid q, x) = P_1(y' \mid q, \overline{x}) = 3/4, P_1(y' \mid \overline{q}, x) = P_1(y' \mid \overline{q}, \overline{x}) = 1/2$. It leads to:

$$Q_1 = \gamma \tfrac{3}{8} \qquad Q_2 = \gamma \left(\tfrac{3}{8} + \tfrac{\epsilon}{56}\right)$$

which are never equal.

case 5: There is a path p connecting Y' and S that goes through an ancestor of both, and contains some $Z_0 \in \mathbf{Z}$.

Let N, Q, R be be defined as in the previous case, also construct G' the same way. Following the same derivation strategy as in case 3, the query expressions become:

$$Q_1 = \gamma \sum_{Z_0} \frac{\sum_Q P_1(y' \mid \mathbf{x}, q) P_1(q) P_1(z_0 \mid q)}{\sum_{Y',Q} P_1(y' \mid \mathbf{x}, q) P_1(q) P_1(z_0 \mid q)} P_1(z_0)$$

$$Q_2 = \gamma \sum_{Z_0} \frac{\sum_Q P_1(y' \mid \mathbf{x}, q) P_1(q) P_1(z_0 \mid q)}{\sum_{Y',Q} P_1(y' \mid \mathbf{x}, q) P_1(q) P_1(z_0 \mid q)} P_1(z_0 \mid S{=}1)$$

Now, $P_1(z_0)$ and $P_1(z_0 \mid S{=}1)$ are derived in similar terms:

$$P_1(z_0) = \sum_Q P_1(z_0, q) = \sum_Q P_1(q) P_1(z_0 \mid q)$$

$$\begin{aligned} P_1(z_0 \mid S{=}1) &= \frac{\sum_R P_1(S=1, z_0, r)}{\sum_{Z_0,R} P_1(S=1, z_0, r)} \\ &= \frac{P_1(z_0) \sum_R P_1(r \mid z_0) P_1(S=1 \mid r)}{\sum_{Z_0} P_1(z_0) \sum_R P_1(r \mid z_0) P_1(S=1 \mid r)} \end{aligned}$$

Use lemma 8 to parametrize $P_1(r \mid z_0) = 1/2 + \epsilon_1/2, P_1(r \mid \overline{z_0}) = 1/2 - \epsilon_1/2, P(z_0 \mid q) = 1/2 + \epsilon_2/2, P(z_0 \mid q) = 1/2 - \epsilon_2/2$ where $\epsilon_i = (1/5)^{k_i}, i = \{1, 2\}$ (using $p = 3/5, q = 2/5$ in both cases). Define $P(S{=}1 \mid r) = 2/3$ and $P(S{=}1 \mid \overline{r}) = 1/2$. Also by lemma 8 let $P_1(y' \mid q, x) = P_1(y' \mid q, \overline{x}) = 3/4, P_1(y' \mid \overline{q}, x) = P_1(y' \mid \overline{q}, \overline{x}) = 1/2$. The queries end up as:

$$Q_1 = \gamma \tfrac{3}{8} \qquad Q_2 = \gamma \left(\tfrac{3}{8} + \tfrac{\epsilon_1 \epsilon_2}{56}\right)$$

Which are never equal.

case 6: There is confounding path between Y' and S consisting of unobservable variables.

The models for this case can be constructed as in case 4, then moving the variables in the in the path from Q to R (included) from the set of observables to the set of unobservables.

Now, suppose condition (d) does not hold. It should be the case that there exists some $X \in \mathbf{X} \cap An_S$ and a $Y' \in \mathbf{Y}$ that are connected through a back-door path p that would be usually blocked by some $Z_0 \in \mathbf{Z}$. There are two possible cases (depicted in Fig. 8) not contradicting previous conditions:

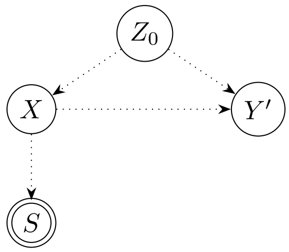
case 1: Z_0 is an ancestor of $\mathbf{X}$ and Y', and S is a descendant of $\mathbf{X}$.

Let $\mathbf{W_1}$ be the nodes in the path between Z_0 and $\mathbf{X}$, $\mathbf{W_2}$ those between Z_0 and Y', and $\mathbf{W_3}$ those between $\mathbf{X}$ and S. As in previous cases, consider the graph G' where all nodes but $\{\mathbf{X}, Z_0, Y', \mathbf{W_1}, \mathbf{W_2}, \mathbf{W_3}\}$ are disconnected from this set. Also, suppose $\mathbf{X}$ and Y' are not connected by any path not going through Z_0. The queries in the corresponding models can be expressed as:

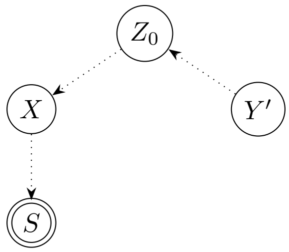
$$\begin{aligned} Q_1 &= \gamma \sum_{Z_0} P_1(y' \mid \mathbf{x}, z_0) P_1(z_0) \\ &= \gamma \sum_{Z_0} P_1(y' \mid z_0) P_1(z_0) = \gamma P_1(y') \\ Q_2 &= \gamma \sum_{Z_0} P_1(y' \mid \mathbf{x}, z_0, S{=}1) P_1(z_0 \mid S{=}1) \\ &= \gamma \sum_{Z_0} P_1(y' \mid z_0) P_1(z_0 \mid S{=}1) \end{aligned}$$

The term $P_1(z_0 \mid S{=}1)$ is available as:

$$P_1(z_0 \mid S{=}1) = \frac{P_1(z_0) \sum_{\mathbf{X}} P_1(\mathbf{x} \mid z_0) P(S{=}1 \mid \mathbf{x})}{\sum_{\mathbf{X},Z_0} P_1(z_0) P_1(\mathbf{x} \mid z_0) P(S{=}1 \mid \mathbf{x})}$$



(a) Case 1



(b) Case 2

Figure 8: Cases considered for the necessity of condition (d) in the proof for Thm. 1. Dotted directed arrows indicate chains of arbitrary length in the graph.

Let $P_1(z_0) = 1/2, P_1(y' \mid z_0) = 1/2 + \epsilon_1/2, P(y' \mid \overline{z_0}) == 1/2 - \epsilon_1/2, P_1(\mathbf{x} \mid z_0) = 1/2 + \epsilon_2/2, P(\mathbf{x} \mid \overline{z_0}) = 1/2 - \epsilon_2/2$. Also $P_1(S{=}1 \mid \mathbf{x}) = 1/2 + \epsilon_3, P_1(S{=}1 \mid \overline{\mathbf{x}}) = 1/2 - \epsilon_3$ where $\epsilon_i = (1/5)^{k_i}, i = 1, 2, 3$ (using lemma 8 with $p = 3/5, q = 2/5$ in all cases):

$$Q_1 = \tfrac{1}{2}\gamma \qquad Q_2 = \gamma\left(\tfrac{1}{2} + \tfrac{\epsilon_1\epsilon_2\epsilon_3}{2}\right)$$

Which are never equal.

case 2: Z_0 is an ancestor of $\mathbf{X}$ and a descendant of Y' and S is a descendant of $\mathbf{X}$.
In this case there are no causal paths between $\mathbf{X}$ and Y' otherwise Z_0 violates condition (a). Lemma 7 can be used to change the direction of the edges in the path from Y' to Z_0 while staying in the same equivalence class, then the same parametrization from the previous case applies.

□

Definition 9 (Generalized Adjustment Criterion Type 2)**.** Let $\mathbf{T}$ be the set of variables measured without selection bias in the overall population. Also, let $\mathbf{X}, \mathbf{Y}, \mathbf{Z}$ be disjoint sets of variables in a causal model with diagram G, augmented with the selection mechanism S, where $\mathbf{Z} \subseteq \mathbf{T}$. Then, $\mathbf{Z}$ satisfies the generalized criterion relative to $\mathbf{X}, \mathbf{Y}$ if:

(a) No element in $\mathbf{Z}$ is a descendant in $G_{\overline{\mathbf{X}}}$ of any $W \notin \mathbf{X}$ which lies on a proper causal path from $\mathbf{X}$ to $\mathbf{Y}$.

(b) All non-causal $\mathbf{X}$-$\mathbf{Y}$ paths in G are blocked by $\mathbf{Z}$.

(c) $\mathbf{Y}$ is independent of the selection mechanism S given $\mathbf{Z}$ and $\mathbf{X}$, i.e. $(\mathbf{Y} \perp\!\!\!\perp S \mid \mathbf{X}, \mathbf{Z})$

Theorem 3 (Generalized Adjustment Formula Type 2)**.** *Let $\mathbf{T}$ is the set of variables measured without selection bias. Given disjoint sets of variables $\mathbf{X}, \mathbf{Y}$ and $\mathbf{Z} \subseteq \mathbf{T}$ and a causal diagram G, then, for every model inducing G, the effect $P(\mathbf{y} \mid do(\mathbf{x}))$ is given by*

$$P(\mathbf{y} \mid do(\mathbf{x})) = \textstyle\sum_{\mathbf{Z}} P(\mathbf{y} \mid \mathbf{x}, \mathbf{z}, S{=}1) P(\mathbf{z}) \qquad (34)$$

if and only if the set $\mathbf{Z}$ satisfies the generalized adjustment criterion type 2 relative to the pair $\mathbf{X}, \mathbf{Y}$.

Proof. (if) Suppose the set $\mathbf{Z}$ satisfies the conditions relative to $\mathbf{X}, \mathbf{Y}$. Then, by conditions (a) and (b), for every model induced by G we have:

$$P(\mathbf{y} \mid do(\mathbf{x})) = \textstyle\sum_{\mathbf{Z}} P(\mathbf{y} \mid \mathbf{x}, \mathbf{z}) P(\mathbf{z})$$

We note that S can be introduced to the first term by cond. (c), which entail Eq. (34).

(Only if) We prove this by the contrapositive: In case conditions (a) or (b) do not hold the same argument as in the proof of the previous type applies here.

To argue the necessity of condition (c), we show that the adjustment formula is not recoverable. Let $\mathbf{V}$ represents all variables in the graph except for the selection mechanism S, we construct two models with distributions P_1 and P_2, compatible with G such that they agree in the probability distribution under selection bias

$$P_1(\mathbf{v} \mid S{=}1) = P_2(\mathbf{v} \mid S{=}1) \qquad (35)$$

and on the non-biased distribution over $\mathbf{Z}$,

$$P_1(\mathbf{z}) = P_2(\mathbf{z}) \qquad (36)$$

but Q_1 in the first model provides a different distribution than Q_2 in the second model. Let P_1 be compatible with G and P_2 compatible with $G_{\overline{S}}$ such that $(\mathbf{V} \perp\!\!\!\perp S)_{P_2}$. Recoverability should hold for any parametrization, hence without loss of generality, we describe Markovian models and assume that every variable is binary. In every construction we parametrize P_1 through its factors and then parametrize P_2 to enforce (35) and (36). Moreover, (35) also equals to $P_2(\mathbf{v})$.

Suppose condition (c) does not hold, then there is an open path between $\mathbf{Y}$ and S not blocked when $\mathbf{Z}$ is observed. As for the previous type proof, we fix any $Y' \in \mathbf{Y}$ not satisfying the condition and consider the query as:

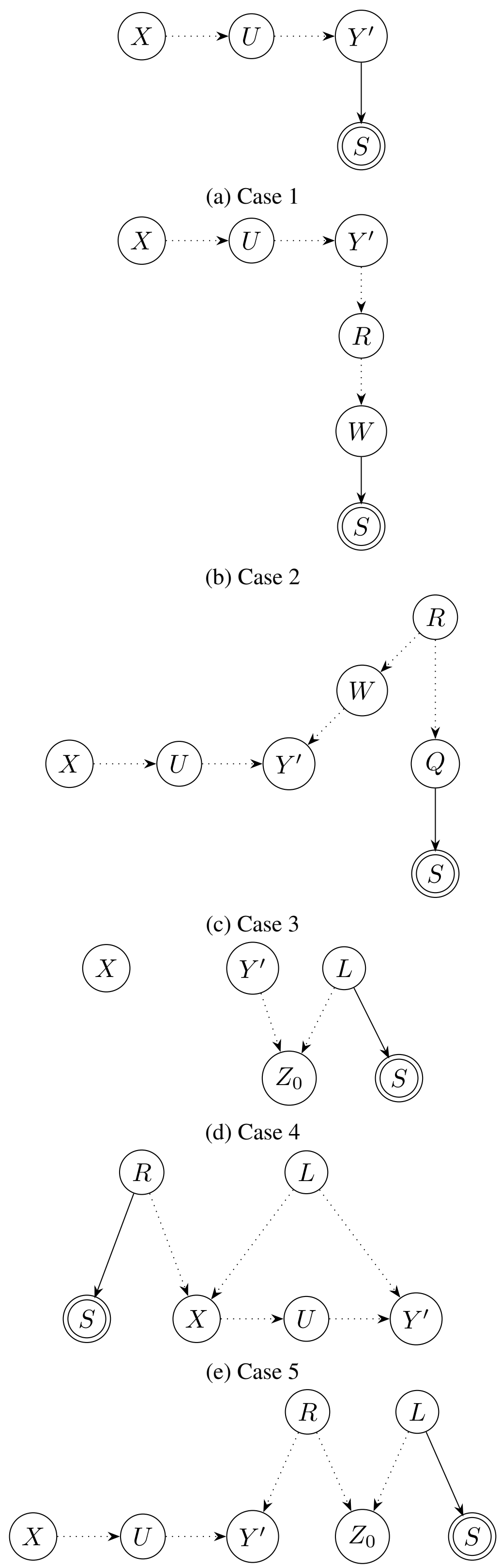
$$Q = \gamma \sum_{\mathbf{Z}} P(y' \mid \mathbf{x}, \mathbf{z}) P(\mathbf{z})$$

where γ represents the product of the marginal distribution of the remaining $\mathbf{Y}$.

Now, let us consider every possible scenario in which condition (c) may be unsatisfied. Fig. 9 illustrates every case for easier reference.

case 1: $Y' \in Pa_S$
Proceed exactly as in case 1 of the proof for Thm. 1.

case 2: There is a directed path p from Y' to S
Consider the same name convention as in case 2 of Thm. 1. It should be the case that $(\{R\} \cup \mathbf{H}) \cap \mathbf{Z} = \emptyset$ for condition (c) to be violated, therefore the same parametrization of the mentioned case applies here.



(a) Case 1
(b) Case 2
(c) Case 3
(d) Case 4
(e) Case 5
(f) Case 6

Figure 9: Ways in which Y' and S can be connected, unidirected dotted paths indicate that it may content an arbitrary number of variables

case 3: The path goes through a common ancestor of Y' and S with no colliders.
This reduces to case 4 in the proof for Thm. 1.

case 4: Non directed path from Y' to S with some $Z_0 \in \mathbf{Z}$ as a collider.
$\mathbf{X}$ must be disconnected from Y' otherwise condition (a) is violated since Z_0 is a descendant of Y' which is in every causal path from $\mathbf{X}$ to Y'. Let L be the common ancestor of S and Z_0, without loss of generality assume that L is directly connected to Z_0, if this were not the case we can apply lemma 8 with $p = 1/2, q = 1/2$ and use the same parametrization below. In this model the causal effect $P(y' \mid do(\mathbf{x})) = P(y')$, however the adjustment formula will not be equal to this effect in every model compatible with the graph. To see this consider a model where every $\mathbf{Z}$ but Z_0 is also disconnected in the graph, we have:

$$\begin{aligned} Q &= \gamma \sum_{\mathbf{Z}} P(y' \mid \mathbf{x}, \mathbf{z}, S{=}1) P(\mathbf{z}) \\ &= \gamma \sum_{Z_0} \sum_{\mathbf{Z}'} P(y' \mid \mathbf{x}, z_0, \mathbf{z}', S{=}1) P(z_0, \mathbf{z}') \\ &= \gamma \sum_{Z_0} P(y' \mid \mathbf{x}, z_0, S{=}1) P(z_0) \\ &= \gamma \sum_{Z_0} \frac{P(y', \mathbf{x}, z_0, S{=}1)}{\sum_{Y'} P(y', \mathbf{x}, z_0, S{=}1)} P(z_0) \end{aligned}$$

We can write the numerator of the fraction in the last expression as:

$$\begin{aligned} &P(y', \mathbf{x}, z_0, S{=}1) \\ &= \sum_{L} P(y', \mathbf{x}, z_0, l, S{=}1) \\ &= P(\mathbf{x}) \sum_{L} P(y') P(l) P(z_0 \mid y', l) P(S{=}1 \mid l) \end{aligned}$$

Let $\alpha_L(y') = P(y')P(l)$, $f(y', l, z_0) = P(z_0 \mid y', l) P(S{=}1 \mid l)$ and $g(y', l, z_0) = P(z_0 \mid y', l)$. We have:

$$\begin{aligned} Q &= \gamma \sum_{Z_0} \frac{P(\mathbf{x}) \sum_L \alpha_L(y') f(y', l, z_0)}{P(\mathbf{x}) \sum_{L,Y'} \alpha_L(y') f(y', l, z_0)} P(z_0) \\ &= \gamma \sum_{Z_0} \frac{\sum_L \alpha_L(y') f(y', l, z_0)}{\sum_{L,Y'} \alpha_L(y') f(y', l, z_0)} P(z_0) \end{aligned}$$

Consider the parametrization:
$P_1(y') = P_1(l) = 1/2$,
$P_1(z_0 \mid y', l) = 1/2 + \epsilon$, $P_1(z_0 \mid y', \bar{l}) = 1/2 - \epsilon$, $P_1(z_0 \mid \overline{y'}, l) = 1/2$, $P_1(z_0 \mid \overline{y'}, \bar{l}) = 1/2$, for $0 < \epsilon < 1/2$.
Let $P_1(S{=}1 \mid l) = \alpha$, $P_1(S{=}1 \mid \bar{l}) = \beta$, and pick any $0 < \alpha, \beta < 1$. We can calculate $P(z_0)$ as:

$$P(z_0) = \sum_{Y',L} P(z_0 \mid y', l) P(y') P(l) = 1/2$$

Computing the queries with the given parametrization we obtain:

$$P(\mathbf{y} \mid do(\mathbf{x})) = \frac{\gamma}{2}$$
$$Q = \frac{\gamma}{2} \frac{(\alpha+\beta)^2 - 2\epsilon^2(\alpha-\beta)^2}{(\alpha+\beta)^2 - \epsilon^2(\alpha-\beta)^2}$$

Q is always different that $P(\mathbf{y} \mid do(\mathbf{x}))$ for $\alpha \neq \beta$.

case 5: There is a path between Y' and S passing by an ancestor of Y' having some $X' \in \mathbf{X}$ as a collider.
Such path would have arrows incoming to X'. But it is also an open non-causal path between Y' and X', violating condition (b).

case 6: There is a path p between Y' and S passing by an ancestor R of Y' having some $Z_0 \in \mathbf{Z}$ as a collider.
Let us consider the variables $\mathbf{X}, Y', R, Z_0, L$, such that L is a common ancestor of Z_0 and S, that together with R has converging arrows into Z_0. The path p can be seen as the concatenation of four segments $p_1, \dots, p_4$ such that p_1 is the segment $R, \dots, Y'$, p_2 is the segment $R, \dots, Z_0$, p_3 is the segment $L, \dots, Z_0$, and p_4 the segment $L, \dots, S$. Note that by construction, there might exist only chains along each of these segments, without loss of generality we assume that those are segments of length one, but it is trivial to stretch those segments using lemma 8 (with p=q=1/2) as in previous cases. When we have multiple Z_0's in p, we will have the concatenation of several segments p_2 and p_3, and it will also be simple to extend the construction. We use a parametrization similar to the previous cases, here $\mathbf{Z}' = \mathbf{Z} \setminus Z_0$:

$$\begin{aligned}
Q_2 &= \gamma \sum_{\mathbf{Z}} P_1(y' \mid \mathbf{x}, \mathbf{z}, S{=}1) P_1(\mathbf{z}) \\
&= \gamma \sum_{Z_0} \sum_{\mathbf{Z}'} P_1(y' \mid \mathbf{x}, z_0, \mathbf{z}', S{=}1) P_1(z_0, \mathbf{z}') \\
&= \gamma \sum_{Z_0} P_1(y' \mid \mathbf{x}, z_0, S{=}1) \sum_{\mathbf{Z}'} P_1(z_0, \mathbf{z}') \\
&= \gamma \sum_{Z_0} P_1(y' \mid \mathbf{x}, z_0, S{=}1) P_1(z_0) \\
&= \gamma \sum_{Z_0} \frac{P_1(y', \mathbf{x}, z_0, S{=}1)}{\sum_{Y'} P_1(y', \mathbf{x}, z_0, S{=}1)} P_1(z_0)
\end{aligned}$$

We can write the numerator of the fraction in the last expression as:

$$\begin{aligned}
&P_1(y', \mathbf{x}, z_0, S{=}1) \\
&= \sum_{R,L} P_1(y', \mathbf{x}, z_0, r, l, S{=}1) \\
&= \sum_{R,L} P_1(\mathbf{x}) P_1(y' \mid r) P_1(r) P_1(z_0 \mid r, l) P_1(S{=}1 \mid l) P_1(l) \\
&= P_1(\mathbf{x}) \sum_R P_1(y' \mid r) P_1(r) \sum_L P_1(z_0 \mid r, l) P_1(S{=}1 \mid l) P_1(l)
\end{aligned}$$

Let $\alpha_R(y') = P_1(y' \mid r) P_1(r)$, $f(r, z_0) = \sum_L P_1(z_0 \mid r, l) P_1(S{=}1 \mid l) P_1(l)$ and $g(r, z_0) = \sum_L P_1(z_0 \mid r, l) P_1(l)$. We have:

$$\begin{aligned}
Q_2 &= \gamma \sum_{Z_0} \frac{P_1(\mathbf{x}) \sum_R \alpha_R(y') f(r, z_0)}{P_1(\mathbf{x}) \sum_{R,Y'} \alpha_R(y') f(r, z_0)} P_1(z_0) \\
&= \gamma \sum_{Z_0} \frac{\sum_R \alpha_R(y') f(r, z_0)}{\sum_{R,Y'} \alpha_R(y') f(r, z_0)} P_1(z_0)
\end{aligned}$$

Consider the following parametrization:
$P_1(r) = P_1(l) = 1/2$, $P_1(y' \mid r) = 1/2 + \epsilon, P_1(y' \mid \overline{r}) = 1/2 - \epsilon$, $P_1(z_0 \mid r, l) = 1/2 + \epsilon, P_1(z_0 \mid r, \bar{l}) = 1/2 - \epsilon, P_1(z_0 \mid \overline{r}, l) = 1/2, P_1(z_0 \mid \overline{r}, \bar{l}) = 1/2$, for $0 < \epsilon < 1/2$.
Let $P_1(S{=}1 \mid l) = \alpha, P_1(S{=}1 \mid \bar{l}) = \beta$, and pick any $0 < \alpha, \beta < 1$. We can calculate $P_1(z_0)$ as:

$$P_1(z_0) = \sum_{R,L} P_1(z_0 \mid r, l) P_1(r) P_1(l) = 1/2$$

Computing the queries with the given parametrization we obtain:

$$Q_1 = \tfrac{\gamma}{2}$$
$$Q_2 = \frac{\gamma}{2}\left(1 - \frac{2\epsilon^3(\alpha-\beta)^2}{(\alpha+\beta)^2 - \epsilon^2(\alpha-\beta)^2}\right)$$

Q_2 is always different than Q_1 for $\alpha \neq \beta$ and $0 < \epsilon < 1/2$.

□