Quantum Voting and Bypassing Gibbard-Satterthwaite's Impossibility Theorem

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Abstract

The Gibbard–Satterthwaite (GS) theorem is a central impossibility result for single-winner voting on ordinal (ranking) ballots: when $|A| > 2$, every onto and strategyproof rule is dictatorial. We revisit this no-go phenomenon under a ballot model motivated by indecisive voters and quantum information: a voter may submit not a single ranking, but a state over rankings (a probability mixture, or more generally a density operator on the permutation basis). This raises the key question: how should "preference" and "manipulation" be interpreted when ballots are uncertain/quantum?

We use a projection-and-trace semantics for pairwise comparisons: a ballot's support for $x > y$ is $Tr(\Pi_{x>y}\rho)$. This yields sharp and unsharp (0/1 and nonzero) preference types and a corresponding "support-flip" notion of strategic manipulation. Within this framework we formulate a natural Quantum GS conjecture and disprove it by counterexample using Quantum Condorcet Voting (QCV) plus a natural winner readout.

Classical Baseline: GS on Point Rankings

Classical GS is a theorem about point ballots: each voter reports one linear order $R_i \in \mathcal{L}(A)$. A social choice rule maps $(R_1, \ldots, R_n)$ to a winner in A, and strategyproofness is defined relative to that point domain. For $|A| > 2$, "onto + strategyproof" forces dictatorship.

Our shift: keep the primitive objects ordinal (rankings) but enlarge the ballot domain from points to distributions/states over $\mathcal{L}(A)$. This isolates a concrete source of brittleness in GS: the theorem does not automatically carry over when ballots encode uncertainty at the level of full rankings and when incentives are lifted using a semantics appropriate to that domain.

Ballot Model: Density Operators over the Ranking Basis

Let $\mathcal{L}(A)$ be the set of linear orders on A. Define the ranking Hilbert space

$$R \cong \mathbb{C}^{|\mathcal{L}(A)|}, \qquad \{\, |R\rangle : R \in \mathcal{L}(A) \,\} \text{ (ranking basis).}$$

A voter's ballot is a density operator $\rho_i \in D(R_i)$. A ballot profile can be taken as a joint state

$$\rho \in D(R_1 \otimes \cdots \otimes R_n),$$

which allows correlated ballots at the level of the formalism.

Indecisive voting as a restriction. If we restrict ballots to classical mixtures over basis rankings (equivalently, diagonal/"classical" structure in the ranking basis), we recover "indecisive ballots": probability distributions over rankings rather than over candidates.

Support Semantics: Projectors for Pairwise Comparisons

For a pair (x, y), let

$$S_{x>y} \coloneqq span\{\, |R\rangle : x \succ_R y \}, \qquad \Pi_{x>y} \text{ projector onto } S_{x>y}.$$

Define the ballot's support for $x > y$ as

$$s_i(x > y) \coloneqq Tr(\Pi_{x>y}\, \rho_i).$$

Interpretation: $s_i(x > y)$ is the probability mass (more generally, the trace weight) that voter i's ballot assigns to rankings where $x > y$.

This induces sharp/unsharp preference types used throughout:

- **Strong +:** $s_i(x > y) = 1$ (ballot supports only $x > y$);
- **Strong −:** $s_i(x > y) = 0$ (ballot forbids $x > y$);
- **Weak:** $s_i(x > y) > 0$ (ballot allows $x > y$).

Onto (the relevant notion here): for each candidate a, there exists a profile for which the social outcome assigns support only to a as winner (so "$|A| > 2$ has bite").

Open directions

First, strengthen incentives beyond the support semantics (e.g., p-threshold preferences or notions of moving the societal ballot "closer" to a voter's ballot) and ask when a GS-style dictatorship reappears. Second, test generality: do other quantum rules (e.g., Quantum Majority Rule) and other welfare-to-choice readouts behave like QCVNE? Third, map the boundary: what domain/axiom restrictions recover the classical regime, and which other theorems (Sen, Müller–Satterthwaite) or non-ranked ballots (e.g., combined approval) admit analogous translations?

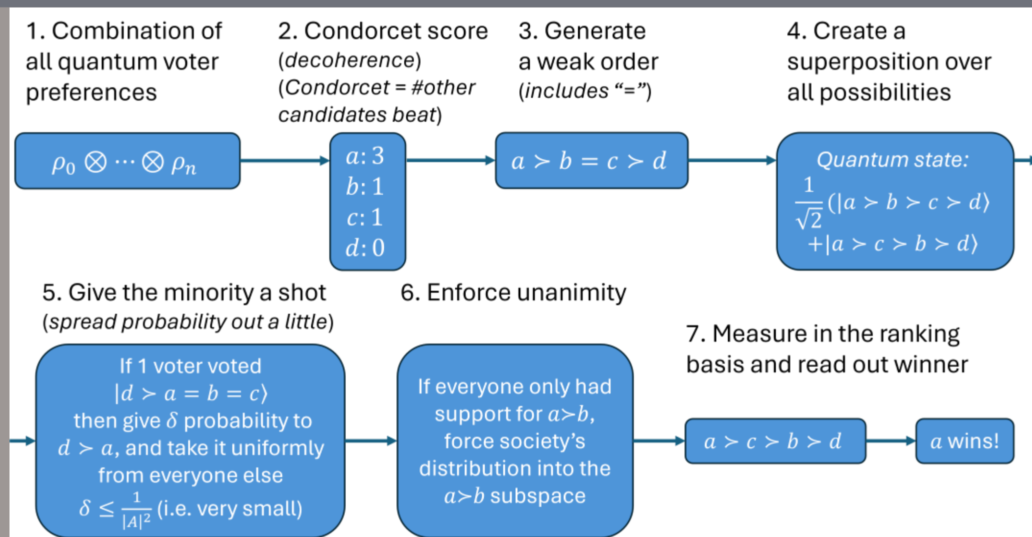

Fig. 1: QCV Pipeline. QCV maps ballots $\rho_1, \ldots, \rho_n$ over the ranking basis to a societal ranking state. Two steps drive the GS counterexample: Step 5 ("minority shot") ensures any pairwise relation with nonzero individual support retains nonzero societal support; Step 6 ("unanimity enforcement") projects unanimously certain relations to certainty at the social level. A natural top-candidate readout then produces a single-winner distribution from societal support on "a is top" subspaces. Theorem X: the resulting rule is onto, QIC (support-flip), and non-dictatorial, refuting QGS for $|A| > 2$.

Incentives: Manipulation as Support Flips

When ballots are uncertain/quantum, the notion of manipulation must specify what counts as a "better" social outcome. We adopt a minimal, qualitative notion aligned with sharp/unsharp semantics.

A voter manipulates if, by misreporting $\rho_i \to \rho_i'$, they can force a socially relevant support flip that improves with respect to their preference type on some pair (x,y):

- **Strong + exploit:** raise society from "not certain" to certain ($< 1 \to 1$);
- **Strong− exploit:** reduce society from "some support" to no support ($> 0 \to 0$);
- **Weak exploit:** create support where there was none ($0 \to > 0$).

A rule is **Quantum Incentive Compatible (QIC)** if no single voter can achieve any beneficial support flip (pairwise, and after winner readout, at the level of winners).

Conjecture: A Quantum Gibbard–Satterthwaite Analogue

Conjecture (Quantum Gibbard–Satterthwaite, QGS). Every QIC voting rule that is onto more than two alternatives ($|A| > 2$) must be a quantum dictatorship (i.e., one voter controls the societal support structure, in the sharp/unsharp sense).

The purpose of QGS is to test whether the GS dictatorship conclusion survives once ballots are states over rankings and incentives are interpreted via support.

Counterexample: Quantum Condorcet Voting

The counterexample comes from Quantum Condorcet Voting (QCV), which is naturally described as a protocol on ranking states. Fig. 1 shows the pipeline, but only two steps matter for incentives:

Minority shot (Step 5): if any voter has $s_i(x > y) > 0$, the societal state is forced to have $Tr(\Pi_{x>y}\rho_{soc}) > 0$.

Unanimity enforcement (Step 6): if all voters have $s_i(x > y) = 1$, the societal state is projected so $Tr(\Pi_{x>y}\rho_{soc}) = 1$.

Any attempted misreport that would change support for a pairwise relation is blocked by either step 5 or 6. To obtain a single-winner rule, apply the natural top-candidate readout. (This is the "Natural Extension" in the paper.)

Theorem 3.4: Disproof of the QGS Conjecture

For $|A| > 2$, the voting rule obtained by QCV plus the natural top-candidate readout is simultaneously:

1. QIC (no beneficial support-flip manipulation),
2. onto, and
3. non-dictatorial (sharp and unsharp).

Hence the Quantum GS conjecture is false. Restricting ballots to classical mixtures yields the same conclusion for indecisive voting.

References & Contact

[1] Bao, N., and Yunger Halpern, N. Quantum voting and violation of arrow's impossibility theorem. *Physical Review A* 95, 6 (Jun 2017)

[2] Sun, X., He, F., Sopek, M., and Guo, M. Schrödinger's ballot: Quantum information and the violation of arrow's impossibility theorem. *Entropy* 23, 8 (2021), 1083.

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