The goals for this lecture are:

- Understand that PageRank is an analytic function of $\alpha$ and what this means.
- Work through the result that PageRank has a unique limit as $\alpha \to 1$.
- Understand how the strong component structure of the web impacts computing PageRank and the choice of $\alpha$.
- See how to compute Personalized PageRank more efficiently.

The PageRank function

$$(I - \alpha P)x = (1 - \alpha)v$$

Considering the behavior as a function of $\alpha$

$$(I - \alpha P)x(\alpha) = (1 - \alpha)v$$

Then $x(\alpha)$ exists if $\alpha \neq \frac{1}{\lambda(P)}$, where $\lambda(P)$ is any eigen value of $P$, except $\alpha = 1$

Otherwise, if $\alpha = \frac{1}{\lambda(P)} = \frac{1}{\lambda^*}$, then there exists $z$ where $Pz = \lambda^*z$
\[(\lambda^* I - P)z = 0\]
\[(I - \frac{1}{\lambda^*} P)z = 0\]
L.H.S is singular \(\rightarrow\) No unique solution.

Check an example of matlab code on course website.

PageRank is a vector analytic function \(f\) of \(\alpha\), \(\alpha \in [-1, 1]\)
\[x(\alpha) = (1 - \alpha) \sum_{k=0}^{\infty} (\alpha P)^k v\]

PageRank is also a rational function. A rational function \(x(\alpha) = \frac{g(\alpha)}{h(\alpha)}\), where \(g, h\) are polynomial functions of \(\alpha\). For example, \(\frac{\alpha^2}{\alpha^3 + 1}\).

If \(Ax = b\), then \(x_i = \frac{\text{det}(A_i)}{\text{det}(A)}\)
So, for pagerank, \((I - \alpha P)x = (1 - \alpha)v\),
\[\frac{\text{det}(A_i)}{\text{det}(A)} = \text{polynomial/polynomial}\]
For more details and an example, check section 2.6 in [1].

**Derivatives**

\[f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}\]
\[x'(\alpha) = \lim_{h \to 0} \frac{x(\alpha+h) - x(\alpha)}{h}\]
\[e^T x'(\alpha) = e^T \lim_{h \to 0} \frac{x(\alpha+h) - x(\alpha)}{h} = \lim_{h \to 0} \frac{e^T x(\alpha+h) - e^T x(\alpha)}{h} = 0\]

\[\frac{d}{dx} f(x)g(x) = f'(x)g(x) + f(x)g'(x)\]
\[\frac{d}{dx}[x(\alpha) = \alpha Px + (1 - \alpha)v]\]
\[x'(\alpha) = \alpha Px' + Px - x\]
\[(I - \alpha P)x'(\alpha) = Px - x\]

sum of R.H.S is 0
Limits

\[ x(\alpha) = (I - \alpha P)^{-1}(1 - \alpha)v \]

Suppose \( P = XDX^{-1} \) is a diagonalization of \( P \)

If \( P = P^T \), then \( P = VDV^T, V^T = V^{-1}, V^TV = I \)

\[ P = XDX^{-1} \]

From Perron–Frobenius theorem

\[
D = \begin{pmatrix}
1 & 1 \\
1 & \lambda_1 \\
& \ddots \\
& & \lambda_k
\end{pmatrix}
\]

\[
(I - \alpha P)x = (1 - \alpha)v
\]

\[
(I - \alpha X \begin{pmatrix} I \\ D_1 \end{pmatrix}) X^{-1}x = (1 - \alpha)v
\]

\[
X^{-1}X(I - \alpha \begin{pmatrix} I \\ D_1 \end{pmatrix}) X^{-1}X = (1 - \alpha) X^{-1}v
\]

\[
(I - \alpha \begin{pmatrix} I \\ D_1 \end{pmatrix})y = (1 - \alpha)w
\]

\[
(I - \alpha I)y_1 = (1 - \alpha)w_1 \implies y_1 = w_1
\]

\[
(I - \alpha D)y_2 = (1 - \alpha)w_2 \implies y_2 = \frac{(1-\alpha)w_2}{(1-\alpha d_{ii})}
\]

\[ \alpha \to 1 : y_1 = w_1, y_2(\alpha) \to 0 \]

References