CS 59000-N · Networks & Matrix Computations

Lecture 16

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The goals for this lecture are:

- Understand that PageRank is an analytic function of α and what this means.
- Work through the result that PageRank has a unique limit as $\alpha \to 1$.
- Understand how the strong component structure of the web impacts computing PageRank and the choice of α.
- See how to compute Personalized PageRank more efficiently.

The PageRank function

$(\boldsymbol{I} - \alpha \boldsymbol{P})\mathbf{x} = (1-\alpha)\mathbf{v}$

Considering the behavior as a function of α

$(\boldsymbol{I} - \alpha \boldsymbol{P})\mathbf{x}(\alpha) = (1-\alpha)\mathbf{v}$

$\mathbf{x}(\alpha)$ exists if $\alpha \neq \frac{1}{\lambda(\boldsymbol{P})}$, where $\lambda(\boldsymbol{P})$ is any eigen value of $\boldsymbol{P}$, except $\alpha = 1$

Otherwise, if $\alpha = \frac{1}{\lambda(\boldsymbol{P})} = \frac{1}{\lambda^*}$,

then there exists $\mathbf{z}$ where $\boldsymbol{P}\mathbf{z} = \lambda^* \mathbf{z}$

$(\lambda^* \boldsymbol{I} - \boldsymbol{P})\mathbf{z} = 0$

$(\boldsymbol{I} - \frac{1}{\lambda^*}\boldsymbol{P})\mathbf{z} = 0$

L.H.S is singular $\rightarrow$ No unique solution.

Check an example of matlab code on course website.

PageRank is a vector analytic function f of α, $\alpha \in [-1, 1]$

$\mathbf{x}(\alpha) = (1-\alpha)\sum\limits_{k=0}^{\infty}(\alpha\boldsymbol{P})^k\mathbf{v}$

PageRank is also a rational function.

A rational function $x(\alpha) = \frac{g(\alpha)}{h(\alpha)}$, where g, h are polynomial functions of α. For example, $\frac{\alpha^2}{\alpha^3+1}$.

If $\boldsymbol{A}\mathbf{x} = \mathbf{b}$, then $\mathbf{x}_i = \frac{det(\boldsymbol{A}_i)}{det(\boldsymbol{A})}$

So, for pagerank, $(\boldsymbol{I} - \alpha\boldsymbol{P})\mathbf{x} = (1-\alpha)\mathbf{v}$,

$\frac{det(\boldsymbol{A}_i)}{det(\boldsymbol{A})} = \frac{polynomial}{polynomial}$

For more details and an example, check section 2.6 in [1].

Derivatives

$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

$\mathbf{x}'(\alpha) = \lim_{h\to 0}\frac{\mathbf{x}(\alpha+h)-\mathbf{x}(\alpha)}{h}$

$e^T\mathbf{x}'(\alpha) = e^T\lim_{h\to 0}\frac{\mathbf{x}(\alpha+h)-\mathbf{x}(\alpha)}{h} = \lim_{h\to 0}\frac{\cancelto{1}{e^T\mathbf{x}(\alpha+h)} - \cancelto{1}{e^T\mathbf{x}(\alpha)}}{h} = 0$

$\frac{d}{dx}f(x)g(x) = f'(x)g(x) + f(x)g'(x)$

$\frac{d}{d\alpha}[\mathbf{x}(\alpha) = \alpha\boldsymbol{P}\mathbf{x}(\alpha) + (1-\alpha)\mathbf{v}]$

$\mathbf{x}'(\alpha) = \alpha\boldsymbol{P}\mathbf{x}'(\alpha) + \boldsymbol{P}\mathbf{x}(\alpha) - \mathbf{x}$

$(\boldsymbol{I} - \alpha\boldsymbol{P})\mathbf{x}'(\alpha) = \underbrace{\boldsymbol{P}\mathbf{x}(\alpha) - \mathbf{x}}_{\text{sum of R.H.S is 0}}$

Limits

$\mathbf{x}(\alpha) = (\boldsymbol{I} - \alpha\boldsymbol{P})^{-1}(1-\alpha)\mathbf{v}$

Suppose $\boldsymbol{P} = \boldsymbol{X}\boldsymbol{D}\boldsymbol{X}^{-1}$ is a diagonalization of $\boldsymbol{P}$

If $\boldsymbol{P} = \boldsymbol{P}^T$, then $\boldsymbol{P} = \boldsymbol{V}\boldsymbol{D}\boldsymbol{V}^T, \boldsymbol{V}^T = \boldsymbol{V}^{-1}, \boldsymbol{V}^T\boldsymbol{V} = \boldsymbol{I}$

$\boldsymbol{P} = \boldsymbol{X}\boldsymbol{D}\boldsymbol{X}^{-1}$

From Perron–Frobenius theorem

$$\boldsymbol{D} = \begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & \lambda_1 & & \\ & & & & \ddots & \\ & & & & & \lambda_k \end{pmatrix} \boldsymbol{D} = \begin{pmatrix} \boldsymbol{I} & \\ & \boldsymbol{D}_1 \end{pmatrix}, \, |\boldsymbol{D}_1| < 1 \, , \, |\lambda_i| < 1$$

$(\boldsymbol{I} - \alpha\boldsymbol{P})\mathbf{x} = (1-\alpha)\mathbf{v}$

$(\boldsymbol{I} - \alpha\boldsymbol{X}\begin{pmatrix} \boldsymbol{I} & \\ & \boldsymbol{D}_1 \end{pmatrix}\boldsymbol{X}^{-1})\mathbf{x} = (1-\alpha)\mathbf{v}$

$\cancelto{\boldsymbol{I}}{\boldsymbol{X}^{-1}\boldsymbol{X}}(\boldsymbol{I} - \alpha\begin{pmatrix} \boldsymbol{I} & \\ & \boldsymbol{D}_1 \end{pmatrix})\underbrace{\boldsymbol{X}^{-1}\mathbf{x}}_{\mathbf{y}} = (1-\alpha)\underbrace{\boldsymbol{X}^{-1}\mathbf{v}}_{\mathbf{w}}$

$(\boldsymbol{I} - \alpha\begin{pmatrix} \boldsymbol{I} & \\ & \boldsymbol{D}_1 \end{pmatrix})\mathbf{y} = (1-\alpha)\mathbf{w}$

$(\boldsymbol{I} - \alpha\boldsymbol{I})\mathbf{y}_1 = (1-\alpha)\mathbf{w}_1 \implies \mathbf{y}_1 = \mathbf{w}_1$

$(\boldsymbol{I} - \alpha\boldsymbol{D})\mathbf{y}_2 = (1-\alpha)\mathbf{w}_2 \implies \mathbf{y}_2 = \frac{(1-\alpha)\mathbf{w}_2}{(1-\alpha d_{ii})}$

$\alpha \to 1 : \mathbf{y}_1 = \mathbf{w}_1, \mathbf{y}_2(\alpha) \to 0$

References

[1] David F. Gleich. *Models and Algorithms for PageRank Sensitivity*. PhD thesis, Stanford University, September 2009.