Lecture 12 notes

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Centrality

Question: How important is a vertex in a graph?

Definition: A structural index (S.I.) of a graph \( G : (V, E) \) is a function \( C : V \to \mathbb{R} \) such that for isomorphic graphs \( G, H, C_G(v) = C_H(\phi(v)) \), where \( \phi(v) \) is the image of \( v \) in \( H \).

Definition (Matrix form): Let \( f : \mathbb{R}^{n \times n} \to \mathbb{R}^n \) be a function on the adjacency matrix \( A \). \( f \) is a structural index if and only if \( P^T f(P A P^T) = f(A) \), where \( P \) is any permutation matrix.

Example of permutation matrices:

If we have \( x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \) and need \( P x = \begin{bmatrix} x_2 \\ x_1 \\ x_3 \\ x_4 \end{bmatrix} \)

By setting \( P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \)

we have \( P x = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_1 \\ x_3 \\ x_4 \end{bmatrix} \)

We also have \( P^T P = I \)

Example of \( P \) applied to adjacency matrix.

Suppose we have a graph \( G \) shown below:

![Graph Image](image-url)

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1Chapter 3-5 of Network Analysis
Its adjacency matrix $A$ is:

$$A = \begin{bmatrix}
0 & 1 & 1 & 0 & 0 & 1 \\
1 & 0 & 1 & 0 & 0 & 0 \\
1 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}$$

Suppose $G$ is relabeled into $H$:

We set the permutation matrix $P$ to be a $6 \times 6$ matrix with 1 at the index $(\text{label}_{\text{new}}, \text{label}_{\text{old}})$, such that $P \mathbf{x}_{\text{old}} = \mathbf{x}_{\text{new}}$ and $P^T \mathbf{x}_{\text{new}} = \mathbf{x}_{\text{old}}$.

$$P = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0
\end{bmatrix}$$

$$PAP^T = (PA)P^T$$

$$\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 \\
1 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0
\end{bmatrix} = P^T \text{ (flipping rows of } A \text{ according to } P)$$

$$\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 \\
1 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0
\end{bmatrix} = \begin{bmatrix}
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0
\end{bmatrix} \text{ (flipping columns according to } P^T)$$

$$\begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0
\end{bmatrix} = \begin{bmatrix}
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0
\end{bmatrix} \text{ (this is the adjacency matrix of } H)$$

From now on, we will use a specific graph $G$ to illustrate different kinds of centralities. $G$ is shown below:
Naming convention: we use $C(x)$ is centrality of vertex $x$ or $C_x$. When we have $C(x) \geq C(y)$ (i.e. $C_x \geq C_y$), we say $x$ is more important than $y$.

Example 1: out-degree and in-degree $d(x)$.

For $G$, we have the degree for each vertex marked as below:

To prove $d(x)$ is an S.I.:

- By isomorphism: obviously $d(x)$ is an S.I. because it is label independent.
- By matrix:

\[
\begin{align*}
    f(A) &= Ae \\
    P^T f(PAP^T) &= P^T(PAP^T)e \\
    &= (P^T P)A(P^T e) \\
    &= IAe \\
    &= Ae \\
    &= f(A)
\end{align*}
\]

Example 2: Eccentricity.

Let $e(u) = \max_v d(u,v)$, where $d(u,v)$ denote the distance between $u$ and $v$.

We have $e_c(u) = \frac{1}{e(u)}$ is an S.I.
Example 3: Closeness / Transmission number.

Let \( t(u) = \sum_v d(u, v) \), where \( d(u, v) \) denote the distance between \( u \) and \( v \).

We have \( t_c(u) = \frac{1}{t(u)} \) is an S.I.

\[
t = 1 \times 4 + 2 \times 4 = 12
\]

Example 4: Betweenness.

\[ b_c(u) = \sum_{s, t \neq u} \frac{\sigma_{st}(u)}{\sigma_{st}}. \sigma_{st} \text{ is the number of shortest paths between vertices } s \text{ and } t. \sigma_{st}(u) \text{ is the number of shortest paths between } s \text{ and } t \text{ that pass } u. \]

Suppose we have a graph shown below, then vertex 4 is the most important vertex according to betweeness.

Example 5: Katz Index.

Consider an adjacency matrix \( A \) for representing a voting result, where if \( A_{ij} = 1 \), we say \( i \) voted for \( j \). \([A^T e]\) is the number of votes for \( j \). Suppose people there were a set of people who voted for \( i \) and then \( i \) voted for \( j \). We wanted to count the votes from all of these people who voted for \( i \) as well. Then the count of votes becomes:

\[
[A^T e] + [(A^T)^2 e]
\]

Then following this logic, why cannot we count the votes in an infinite order:

\[
[A^T e] + [(A^T)^2 e] + \cdots + [(A^T)^k e] + \cdots
\]

This scheme has a problem that it’ll generate infinite counts. We can modify the counting scheme a little bit by dampening the weight of vote as the order becomes higher. This is accomplished by multiplying \( \alpha \) to \( A \) where \( 0 < \alpha < 1 \). Then we have the count of votes as:

\[
[\alpha A^T e] + [\alpha(A^T)^2 e] + \cdots + [\alpha(A^T)^k e] + \cdots
\]

Katz index is then defined as:

\[
k = \sum_{i=1}^{\infty} (\alpha A^T)^i e
\]
When $A$ is 1-by-1 matrix (i.e. scalar 1), $k_1 = 1 + \alpha + \alpha^2 + \cdots = \frac{1}{1-\alpha}$, if $|\alpha| < 1$. That is, $k_1$ is a geometric series.

If we generalize geometric series to matrices, we have the Neumann series, which is named after Carl Gottfried Neumann.

**Neumann series:** $\sum_{l=0}^{\infty} A^l \rightarrow (I - A)^{-1}$ if $\rho(A) < 1$. $\rho(A)$ is the spectral radius of $A$. (recall $\rho(A) = \max_i (|\lambda_i|)$).

Then we can write Katz index as:

$$k = ((I - \alpha A^T)^{-1} - I)e, \text{ if } \rho(\alpha A^T) < 1$$

$$(I - \alpha A^T)k = (I - (I - \alpha A^T))e$$

$$(I - \alpha A^T)k = \alpha A^Te$$

We can solve this linear system to get Katz index $k$. The Richardson method gives:

$$(I - \alpha A^T)x = \alpha A^Te$$

$$r^{(t)} = f - (I - \alpha A^T)x^{(t)}$$

$$r^{(t)} = f - x^{(t)} + \alpha A^Tx^{(t)}$$

$$x^{(t+1)} = x^{(t)} + r^{(t)}$$

$$x^{(t+1)} = f + \alpha A^Tx^{(t)}$$

Let $k^{(t)} = \sum_{i=1}^{t} (\alpha A^T)^i e$. In the homework, we’ll see that $x^{(t)} = k^{(t)}$, i.e. the Richardson method produces a truncated sum of the Neumann series.