LECTURE NOTES
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Lecture 12 notes

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Centrality[1]

Question: *How important is a vertex in a graph?*

Definition: A structural index (S.I.) of a graph $G : (V, E)$ is a function $C : V \to \mathbb{R}$ such that for isomorphic graphs G, H, $C_G(v) = C_H(\phi(v))$, where $\phi(v)$ is the image of v in H.

Definition (Matrix form): Let $f : \mathbb{R}^{n \times n} \to \mathbb{R}^n$ be a function on the adjacency matrix $\boldsymbol{A}$. f is a structural index if and only if $\boldsymbol{P}^T f(\boldsymbol{P}\boldsymbol{A}\boldsymbol{P}^T) = f(\boldsymbol{A})$, where $\boldsymbol{P}$ is any permutation matrix.

Example of permutation matrices:

$$\text{If we have } \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \text{ and need } \boldsymbol{P}\mathbf{x} = \begin{bmatrix} x_2 \\ x_1 \\ x_3 \\ x_4 \end{bmatrix}$$

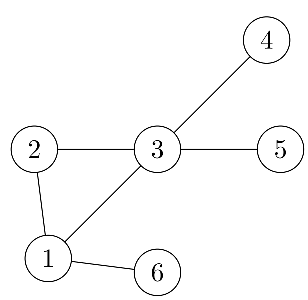
$$\text{By setting } \boldsymbol{P} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\text{we have } \boldsymbol{P}\mathbf{x} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_1 \\ x_3 \\ x_4 \end{bmatrix}$$

$$\text{We also have } \boldsymbol{P}^T\boldsymbol{P} = \boldsymbol{I}$$

Example of $\boldsymbol{P}$ applied to adjacency matrix.

Suppose we have a graph G shown below:

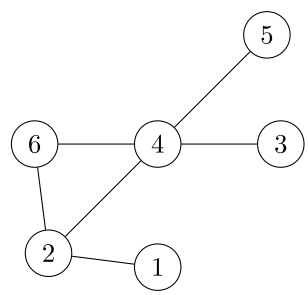


[1] Chapter 3-5 of Network Analysis

Its adjacency matrix $\boldsymbol{A}$ is:

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Suppose G is relabeled into H:

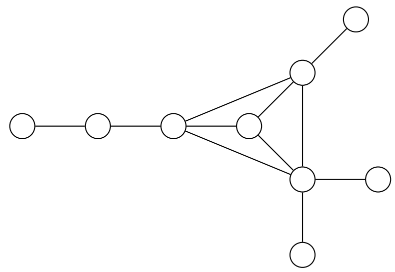


We set the permutation matrix $\boldsymbol{P}$ to be a 6×6 matrix with 1 at the index $(label_{new}, label_{old})$, such that $\boldsymbol{P}\mathbf{x}_{old} = \mathbf{x}_{new}$ and $\boldsymbol{P}^T\mathbf{x}_{new} = \mathbf{x}_{old}$.

$$\boldsymbol{P} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} \boldsymbol{PAP}^T &= (\boldsymbol{PA})\boldsymbol{P}^T \\ &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \boldsymbol{P}^T \text{ (flipping rows of } \boldsymbol{A} \text{ according to } \boldsymbol{P}\text{)} \\ &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \text{ (flipping columns according to } \boldsymbol{P}^T\text{)} \\ &= \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \end{bmatrix} \text{ (this is the adjacency matrix of } H\text{)} \end{aligned}$$

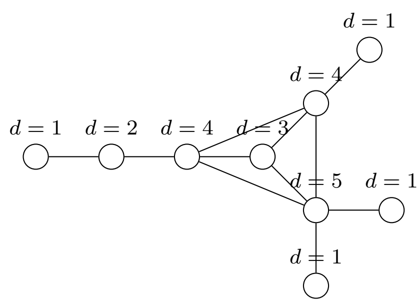
From now on, we will use a specific graph $\mathcal{G}$ to illustrate different kinds of centralities. $\mathcal{G}$ is shown below:



Naming convention: we use $C(x)$ is centrality of vertex x or C_x. When we have $C(x) \geq C(y)$ (i.e. $C_x \geq C_y$), we say x is more important than y.

Example 1: out-degree and in-degree $d(x)$.

For $\mathcal{G}$, we have the degree for each vertex marked as below:



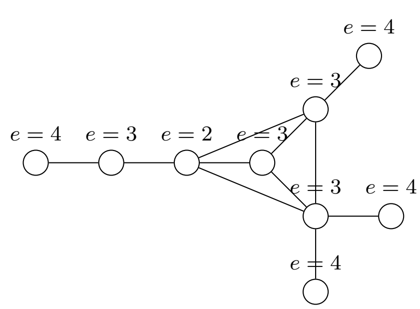
To prove $d(x)$ is an S.I.:

- By isomorphism: obviously $d(x)$ is an S.I. because it is label independent.
- By matrix:

$$\begin{aligned} f(\boldsymbol{A}) &= \boldsymbol{A}\mathbf{e} \\ \boldsymbol{P}^T f(\boldsymbol{P}\boldsymbol{A}\boldsymbol{P}^T) &= \boldsymbol{P}^T(\boldsymbol{P}\boldsymbol{A}\boldsymbol{P}^T)\mathbf{e} \\ &= (\boldsymbol{P}^T\boldsymbol{P})\boldsymbol{A}(\boldsymbol{P}^T\mathbf{e}) \\ &= \boldsymbol{I}\boldsymbol{A}\mathbf{e} \\ &= \boldsymbol{A}\mathbf{e} \\ &= f(\boldsymbol{A}) \end{aligned}$$

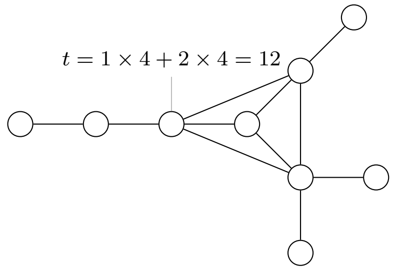
Example 2: Eccentricity.

Let $e(u) = \max\limits_{v} d(u,v)$, where $d(u,v)$ denote the distance between u and v. We have $e_c(u) = \frac{1}{e(u)}$ is an S.I.



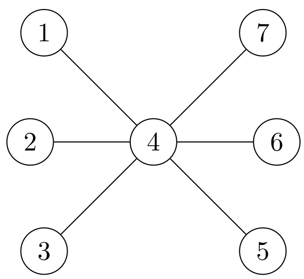
Example 3: Closeness / Transmission number.

Let $t(u) = \sum_{v} d(u, v)$, where $d(u, v)$ denote the distance between u and v. We have $t_c(u) = \frac{1}{t(u)}$ is an S.I.



Example 4: Betweeness.

$b_c(u) = \sum_{s,t \neq u} \frac{\sigma_{st}(u)}{\sigma_{st}}$. σ_{st} is the number of shortest paths between vertices s and t. $\sigma_{st}(u)$ is the number of shortest paths between s and t that pass u. Suppose we have a graph shown below, then vertex 4 is the most important vertex according to betweeness.



Example 5: Katz Index.

Consider an adjacency matrix $\boldsymbol{A}$ for representing a voting result, where if $\boldsymbol{A}_{ij} = 1$, we say i voted for j. $[\boldsymbol{A}^T \mathbf{e}]$ is the number of votes for j. Suppose people there were a set of people who voted for i and then i voted for j. We wanted to count the votes from all of these people who voted for i as well. Then the count of votes becomes:

$$\left[\boldsymbol{A}^T \mathbf{e}\right] + \left[(\boldsymbol{A}^T)^2 \mathbf{e}\right]$$

Then following this logic, why cannot we count the votes in an infinite order:

$$\left[\boldsymbol{A}^T \mathbf{e}\right] + \left[(\boldsymbol{A}^T)^2 \mathbf{e}\right] + \cdots + \left[(\boldsymbol{A}^T)^k \mathbf{e}\right] + \cdots$$

This scheme has a problem that it'll generate infinite counts. We can modify the counting scheme a little bit by dampening the weight of vote as the order becomes higher. This is accomplished by multiplying α to $\boldsymbol{A}$ where $0 < \alpha < 1$. Then we have the count of votes as:

$$\left[\alpha \boldsymbol{A}^T \mathbf{e}\right] + \left[(\alpha \boldsymbol{A}^T)^2 \mathbf{e}\right] + \cdots + \left[(\alpha \boldsymbol{A}^T)^k \mathbf{e}\right] + \cdots$$

Katz index is then defined as:

$$\mathbf{k} = \sum_{l=1}^{\infty} (\alpha \boldsymbol{A}^T)^l \mathbf{e}$$

When $\boldsymbol{A}$ is 1-by-1 matrix (i.e. scalar 1), $k_1 = 1 + \alpha + \alpha^2 + \cdots = \frac{1}{1-\alpha}$, if $|\alpha| < 1$. That is, k_1 is a geometric series.

If we generalize geometric series to matrices, we have the Neumann series, which is named after Carl Gottfried Neumann.

Neumann series: $\sum\limits_{l=0}^{\infty} \boldsymbol{A}^l \to (\boldsymbol{I} - \boldsymbol{A})^{-1}$ if $\rho(\boldsymbol{A}) < 1$. $\rho(\boldsymbol{A})$ is the spectral radius of $\boldsymbol{A}$. (recall $\rho(\boldsymbol{A}) = \max\limits_i(|\lambda_i|)$).

Then we can write Katz index as:

$$\mathbf{k} = ((\boldsymbol{I} - \alpha \boldsymbol{A}^T)^{-1} - \boldsymbol{I})\mathbf{e}, \text{ if } \rho(\alpha \boldsymbol{A}^T) < 1$$

$$\begin{aligned}(\boldsymbol{I} - \alpha \boldsymbol{A}^T)\mathbf{k} &= (\boldsymbol{I} - (\boldsymbol{I} - \alpha \boldsymbol{A}^T))\mathbf{e} \\ (\boldsymbol{I} - \alpha \boldsymbol{A}^T)\mathbf{k} &= \alpha \boldsymbol{A}^T \mathbf{e}\end{aligned}$$

We can solve this linear system to get Katz index $\mathbf{k}$. The Richardson method gives:

$$\begin{aligned}(\boldsymbol{I} - \alpha \boldsymbol{A}^T)\mathbf{x} &= \alpha \boldsymbol{A}^T \mathbf{e} \\ \mathbf{r}^{(t)} &= \mathbf{f} - (\boldsymbol{I} - \alpha \boldsymbol{A}^T)\mathbf{x}^{(t)} \\ \mathbf{r}^{(t)} &= \mathbf{f} - \mathbf{x}^{(t)} + \alpha \boldsymbol{A}^T \mathbf{x}^{(t)} \\ \mathbf{x}^{(t+1)} &= \mathbf{x}^{(t)} + \mathbf{r}^{(t)} \\ &= \mathbf{f} + \alpha \boldsymbol{A}^T \mathbf{x}^{(t)}\end{aligned}$$

Let $\mathbf{k}^{(\ell)} = \sum_{i=1}^{\ell} (\alpha \boldsymbol{A}^T)^i \mathbf{e}$. In the homework, we'll see that $\mathbf{x}^{(t)} = \mathbf{k}^{(t)}$, i.e. the Richardson method produces a truncated sum of the Neumann series.