

# OPTIMALITY AND DUALITY FOR LINEAR PROGRAMS

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Recall the standard form for a linear program:

$$\begin{aligned} & \text{minimize} && \mathbf{c}^T \mathbf{x} \\ & \text{subject to} && \mathbf{Ax} = \mathbf{b} \\ & && \mathbf{x} \geq 0. \end{aligned}$$

The material here is from Chapter 13 in Nocedal and Wright, but some of the geometry comes from Griva, Sofer, and Nash.

## 1 THE DUAL OF AN LP

We'll revisit the dual later. I won't have enough time to talk about it in this lecture. But for reference, my favorite way of defining the dual is via the "simple LP"

$$\begin{aligned} & \underset{\mathbf{x}}{\text{minimize}} && \mathbf{c}^T \mathbf{x} \\ & \text{subject to} && \mathbf{Ax} \leq \mathbf{b}. \end{aligned}$$

Its Lagrangian is:

$$\mathcal{L}(\mathbf{x}, \boldsymbol{\lambda}) = \mathbf{c}^T \mathbf{x} - \boldsymbol{\lambda}^T (\mathbf{b} - \mathbf{Ax}),$$

which can be "transposed" to yield:

$$\mathcal{L}(\mathbf{x}, \boldsymbol{\lambda}) = \mathbf{x}^T \mathbf{c} + \mathbf{x}^T \mathbf{A}^T \boldsymbol{\lambda} - \mathbf{b}^T \boldsymbol{\lambda} = (-\mathbf{b})^T \boldsymbol{\lambda} - \mathbf{x}^T (\mathbf{A}^T \boldsymbol{\lambda} - \mathbf{c})$$

which is the Lagrangian of

$$\begin{aligned} & \underset{\boldsymbol{\lambda}}{\text{minimize}} && -\mathbf{b}^T \boldsymbol{\lambda} \\ & \text{subject to} && \mathbf{c} = \mathbf{A}^T \boldsymbol{\lambda}, \boldsymbol{\lambda} \geq 0 \end{aligned}$$

where  $\mathbf{x}$  are now the Lagrange multipliers.

The Lagrange multipliers  $\boldsymbol{\lambda}$  are often called *dual variables* for this reason.

## 2 KKT CONDITIONS ARE NECESSARY AND SUFFICIENT

Let  $\boldsymbol{\lambda}$  be the Lagrange multipliers for the equality constraints and  $\mathbf{s}$  be the multipliers for the inequality constraints. Then the KKT conditions for the primal LP are:

$$\begin{aligned} \mathbf{A}^T \boldsymbol{\lambda} + \mathbf{s} &= \mathbf{c} \\ \mathbf{Ax} &= \mathbf{b} \\ \mathbf{x} &\geq 0 \\ \mathbf{s} &\geq 0 \\ \mathbf{x}^T \mathbf{s} &= 0. \end{aligned}$$

For the rest of the course, you might want to commit these conditions to memory! They'll be very important.

In general, the KKT conditions are only the necessary conditions for a local minimum. However, for an LP, we'll show that they are also sufficient. In other words, any point  $\mathbf{x}$  that satisfies these conditions is a solution, that is, a local minimizer and a global minimizer.

First, note that for any solution  $(\mathbf{x}^*, \boldsymbol{\lambda}^*, \mathbf{s}^*)$  that satisfies the KKT conditions, we have that

$$\mathbf{c}^T \mathbf{x}^* = \mathbf{b}^T \boldsymbol{\lambda}^*.$$

**Quiz** Show this using the KKT conditions!

Now, consider any other feasible point  $\mathbf{f}$  where  $\mathbf{A}\mathbf{f} = \mathbf{b}$ ,  $\mathbf{f} \geq 0$ . We can show that  $\mathbf{c}^T \mathbf{f} \geq \mathbf{c}^T \mathbf{x}^*$  directly:

$$\mathbf{c}^T \mathbf{f} = (\mathbf{A}^T \boldsymbol{\lambda}^* + \mathbf{s}^*)^T \mathbf{f} = \mathbf{b}^T \boldsymbol{\lambda}^* + \mathbf{f}^T \mathbf{s}^* \geq \mathbf{b}^T \boldsymbol{\lambda}^* = \mathbf{c}^T \mathbf{x}^*$$

because  $\mathbf{f} \geq 0$  and  $\mathbf{s}^* \geq 0$ .