

Please answer the following questions in complete sentences in a clearly prepared manuscript and submit the solution by the due date on Gradescope.

Remember that this is a graduate class. There may be elements of the problem statements that require you to fill in appropriate assumptions. You are also responsible for determining what evidence to include. An answer alone is rarely sufficient, but neither is an overly verbose description required. Use your judgement to focus your discussion on the most interesting pieces. The answer to “should I include ‘something’ in my solution?” will almost always be: Yes, if you think it helps support your answer.

Problem 0: Homework checklist

- Please identify anyone, whether or not they are in the class, with whom you discussed your homework. This problem is worth 1 point, but on a multiplicative scale.
- Make sure you have included your source-code and prepared your solution according to the most recent Edstem note on homework submissions.

Problem 1: Steepest descent

1. (Nocedal and Wright, Exercise 3.6) Let’s conclude with a quick problem to show that steepest descent can converge very rapidly! Consider the steepest descent method with exact line search for the function $f(\mathbf{x}) = (1/2) \mathbf{x}^T \mathbf{Q} \mathbf{x} - \mathbf{x}^T \mathbf{b}$.
- $\mathbf{x}^T \mathbf{b}$. Suppose that we know $\mathbf{x}_0 - \mathbf{x}^*$ is parallel to an eigenvector of \mathbf{Q} . Show that the method will converge in a single iteration.
2. Consider the problem

$$\begin{aligned} & \text{minimize} && (1/2) \mathbf{x}^T \mathbf{Q} \mathbf{x} - \mathbf{x}^T \mathbf{b} \\ & \text{subject to} && \mathbf{x} \geq 0 \end{aligned}$$

where we approximate the inequality constraint with a log-barrier term

$$\text{minimize} \quad (1/2) \mathbf{x}^T \mathbf{Q} \mathbf{x} - \mathbf{x}^T \mathbf{b} - \gamma \sum_i \log(x_i).$$

Can we do an exact line search on this problem (assuming that $\mathbf{x} > 0$ for the initial iteration?) If so, give a procedure to do it. If not, give an argument for why not (no formal proof needed).

Problem 2: LPs in Standard Form

Show that we can solve:

$$\text{minimize} \quad \sum_i \max(\mathbf{a}_i^T \mathbf{x} - b_i, 0)$$

by constructing an LP in standard form.

Problem 3: Duality

Show that these two problems are dual by showing the equivalence of the KKT conditions:

$$\begin{array}{ll} \underset{\mathbf{x}}{\text{minimize}} & \mathbf{c}^T \mathbf{x} \\ \text{subject to} & \mathbf{A}\mathbf{x} \geq \mathbf{b}, \mathbf{x} \geq 0 \end{array} \quad \text{and} \quad \begin{array}{ll} \underset{\boldsymbol{\lambda}}{\text{maximize}} & \mathbf{b}^T \boldsymbol{\lambda} \\ \text{subject to} & \mathbf{A}^T \boldsymbol{\lambda} \leq \mathbf{c}, \boldsymbol{\lambda} \geq 0 \end{array} .$$

Problem 4: Geometry of LPs

(Griva, Sofer, and Nash, Problem 3.12) Consider the system of constraints $\mathbf{A}\mathbf{x} = \mathbf{b}, \mathbf{x} \geq 0$ with

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 4 & 5 & 6 & 0 & 1 & 0 \\ 7 & 8 & 9 & 0 & 0 & 1 \end{bmatrix}, \text{ and } \mathbf{b} = \begin{bmatrix} 12 \\ 15 \\ 18 \end{bmatrix}$$

Is $\mathbf{x} = [1 \ 1 \ 1 \ 0 \ 0 \ 0]^T$ a basic feasible point? Explain your answer precisely in terms of the definition.

Problem 5: Using the geometry

(Griva, Sofer, and Nash, Section 4.3, problem 3.13.) Suppose that a linear program originally included a free variable x_i where there were no upper-and-lower bounds on its values. As we described in class, this can be converted into a pair of variables x_i^+ and x_i^- such that $x_i^+, x_i^- \geq 0$ and x_i is replaced with the difference $x_i^+ - x_i^-$. Prove that a basic feasible point can have only one of x_i^+ or x_i^- different from zero. (Hint: this is basically a one-line proof once you see the right characterization. I would suggest trying an example.)

(Optional) Problem 6: Constraint elimination

Work with an LLM to cook up an algorithm for redundant constraint elimination. See what it suggests and ask it to give you references to known algorithms or books, etc. so that you can verify what it produces. Test the algorithm on a collection of real world LPs.