Lecture 24

- Alternating Optimization ✓
- "Quadratic Programming"

↳ Types of QPs!

"Easy + More interesting"

↳ Active Set method for QPs

→ like Simplex.

QPs

$$\min \ \tfrac{1}{2}\underline{x}^T G \underline{x} + \underline{x}^T \underline{c}$$

$$\text{s.t.} \quad A\underline{x} = b \quad .$$

$$C\underline{x} \geq \underline{d} \quad .$$

Cases:

$G \succeq 0$ (convex)

"Easy"

G is indef. $\quad$ (neg. def.) $G \preceq 0$

↳ Non-convex

↳ NP-hard ☹

$W =$

W

$$\max \ \frac{1}{2}x^TAx$$
$$\text{st} \quad x \geq 0$$
$$e^Tx = 1$$

~ max clique.

Examples of QPs

Portfolio Optimization

If I can invest in a Set of Stocks w/ returns $r_1, \ldots, r_n$

then I want to

Var/ Corr. among Stocks.

$$\max \ x^Tr - K[x^TGx]$$
$$\text{s.t.} \quad x_i \geq 0, \ \sum x_i = 1.$$

SVM

min

st.

SQP -

Given problem

min

s.t.

Create a approx fo

$\frac{1}{2} \underline{w}^T \underline{w}$

$[X^T \; I] \begin{bmatrix} \underline{w} \\ \underline{b} \end{bmatrix} \geq \underline{e}$

Seq. quad. prog.

a general non-linear

$f(x)$

$\underline{E}(\underline{x}) = \underline{b}$

$I(\underline{x}) \geq 0$

Series of QP

Min $\frac{1}{2} \underline{p}^T H \underline{p} + g^T p$

Hessian + gradient

Surrogate

p = Step.

s.t. $J_B^T \underline{p} + \underline{E}(\underline{x}_c) = 0$

$J_I^T \underline{p} + I(x_c) = 0$.

Min

s.t.

See

$$\underline{x}^T L \underline{x} + \underline{x}^T \underline{s}$$

$x_s = 1, \; x_t = 0$

$x \geq 0$

my Research!

How to Solve?

Case 1: Eq. Constraints only!

$$\min \; \frac{1}{2} \underline{x}^T G \underline{x} + \underline{x}^T \underline{c}$$

$$\text{s.t.} \quad A\underline{x} = \underline{b}$$

Hint! you've seen this before!

Quiz today!

How to

L-word

N-word.

$\mathcal{L}(x, \lambda)$

$$\frac{\partial \mathcal{L}}{\partial x} = 0$$

Solve

- Lagrangian

- Null-space.

$= f(x) - \lambda^T(Ax-b)$

$= \frac{1}{2}x^T G x + x^T c - \lambda^T(Ax-b)$

$$\Rightarrow \begin{bmatrix} Gx + c - A^T\lambda = 0 \\ Ax = b \end{bmatrix}$$

$$\Updownarrow$$

$$\begin{bmatrix} G & A^T \\ A & 0 \end{bmatrix}\begin{bmatrix} x \\ -\lambda \end{bmatrix} = \begin{bmatrix} -c \\ b \end{bmatrix}$$

For a few reasons, we want to solve for

$$\underset{\text{"guess"}}{\underset{\uparrow}{x_0}} + \underset{\text{step}}{\underset{\uparrow}{p}} = \underset{\text{soln.}}{\underset{\uparrow}{x}}$$

$$\Downarrow$$

QPE

$$\begin{bmatrix} G & -A^T \\ A & 0 \end{bmatrix}\begin{bmatrix} p \\ \lambda \end{bmatrix} = \begin{bmatrix} c + Gx \\ Ax - b \end{bmatrix}$$

(Step version)

Some theory, if

A has full row-rank
and $z^T G z > 0$
then there is a unique
soln.

(Lemma 16.1, Thm. 16.2
in Nocedal + Wright)

How to solve QPE?

Indefinite Saddlept.
Problem

↳ Book: 8 pages.

Article by Benzi, Golub,
Liesen

↳ 131 pages on QPE.

Case 2

$$\min \frac{1}{2} x^T G x + x$$

$$\text{s.t. } Ax = b, Bx$$

KKT gives Eq.

- $Gx + c = A^T \lambda +$
- $\mu^T (Bx - d) = 0$
- $\mu \geq 0$
- $Ax = b$
- $Bx \geq d$

$c^T \underline{c}$

$\geq \underline{d}$

Ineq.

$\downarrow$

$\underline{}$

$B^T \underline{\mu}$

These can be hard if

$G \not\succ 0$

C Thm (16.4).

Idea to solve

Guess the active set of constraints

Then check.

Given $\underline{x}, W \leftarrow$ Working set.

Check KKT.

See which Lagrange multi are neg.

↳ update W.

The general procedure

Given x_k, W_k

$$\begin{tikzcd} x_k \arrow[d] & W_k \arrow[d] \\ \underline{x} & W \end{tikzcd}$$

Solve $\min \frac{1}{2} x^T G x + c^T \underline{x}$

st. $A\underline{x} = \underline{b}$, $B_W x = d_W \leftarrow$

$\Updownarrow$ as a step

$\swarrow (Gx + c)$

$\min \frac{1}{2} p^T G p + g_k^T p$

st $\begin{bmatrix} A \\ B_W \end{bmatrix} p = 0.$

After Solve

if $p \neq 0$

$\downarrow$

Then look for

$x_k + \alpha p.$

for $\alpha = 1$.

Use $\alpha = 1$, or the smallest α s.t. all of the constraints stay feasible.

→ Add new constraint.

($\alpha_k = 0$ is possible!)

if $p = 0$

↳ We may be at soln!

⇒ Check KKT.

$$\underline{G}\underline{x} + \underline{c} = -A^T\lambda + B_W^T\mu_W$$

Get from QPB.

Assume $\mu_{\neg W} = 0$ (i.e. all non-active are 0.)

If any $\mu_W < 0$, ⇒ remove from W.

if $\mu_W \geq 0$, then —

DONE!

Solved KKT!