Lecture 25

Sherman-Morrison-Woodbury Formula for rank-k updates

Nonlinear equations

- • Jacobians
- • Newton's equation
- • Merit functions
- • Newton-Krylov method
- • Homotopy methods

Rank-K updates to a matrix or a linear system.

$$A \longrightarrow A + \underline{u}\underline{v}^T$$ (ran

$$\longrightarrow A + UV^T$$ (rank

$A: n \times n$, $U: n \times k$, $V: n \times k$.

How does A^{-1} change?

→ A^{-1} means solve a system of Equation, not actually compute it..

→ We have some known / fast way to do A^{-1} and we want $(A + UV^T)^{-1}$.

k 1)

k)

Goal:

$$(A+uv^T)^{-1} = \boxed{\quad ? \quad}$$

Case 1: Solve for x in a rank 1 update to $A\underline{x}=\underline{b}$.

$$(A+\underline{u}\,\underline{v}^T)\underline{x}=\underline{b}$$

$$A\underline{x}+uv^T\underline{x}=\underline{b}$$

$$\underline{x} = \underbrace{A^{-1}\underline{b}}_{\downarrow \; y} - \underbrace{A^{-1}\underline{u}}_{\downarrow \; \underline{z}}\,\underbrace{\underline{v}^T\underline{x}}_{\gamma(\underline{x})}$$

$$x = y - \gamma(\underline{x})\,\underline{z}$$

$$\gamma(\underline{x}) = v^T\underline{x} = \underline{v}^T y - \gamma(x)\,\underline{v}^T\underline{z}$$

$$\gamma(\underline{x}) = \frac{v^T y}{1+\underline{v}^T\underline{z}}$$

Alg:
1. Solve $Ay=\underline{b}$
2. Solve $A\underline{z}=\underline{u}$
3. Compute $\gamma = \dfrac{v^Ty}{1+\underline{v}^T\underline{z}}$
4. Output $\underline{x}=y-\gamma\,\underline{z}$.

How to get X^{-1} ?

$$(A+uv^T)X = I$$

$$AX + \underline{u}\underline{v}^T X = I$$

$$X = A^{-1} - A^{-1}\underline{u}\underbrace{v^T X}_{g^T}$$

$$g^T = \underline{v}^T X = \underline{v}^T A^{-1} - \underbrace{\underline{v}^T A^{-1}\underline{u}}_{\text{Scalar.}}\, g^T$$

$$g^T + \underline{v}^T A \underline{u}\, g^T = v^T A^{-1}$$

$$(\cancel{I} + v^T A u)\, g^T = v^T A^{-1}$$

$$g^T = \frac{v^T A^{-1}}{1+v^T A u}$$

$$(A+uv^T)^{-1} = \frac{A^{-1} - A^{-1}uv^TA^{-1}}{1+\underline{v}^T A^{-1}\underline{u}}$$

Rank K : generaliz.

$$A + UV^T$$

$$(A+UV^T)X = I$$

$$AX + UV^TX = I$$

$$X = A^{-1} - A^{-1}U\underbrace{V^TX}_{G^T}$$

$$G^T = V^TX = V^TA^{-1} - \underbrace{V^TA^{-1}U}_{K \times K \text{ Matrix}}G^T$$

g - Scalar

$\downarrow$

g - vector

$\downarrow$

G - Matrix

$$(I + V^TA^{-1}U)G^T = V^TA^{-1}$$

$$G^T = (I + V^TA^{-1}U)^{-1}V^TA^{-1}$$

$$(A+UV^T)^{-1} =$$

$$A^{-1} + A^{-1}U(I + U^TA^{-1}V)^{-}$$

$U^T A^{-1}$

Nonlinear Equations

$F(\underline{x}) = 0$

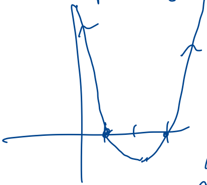
$F: \mathbb{R}^n \to \mathbb{R}^m$

$x: \mathbb{R}^n$

find a zero or root of the Equation $F(\underline{x}) = 0$.

Why called roots? They generalize polynomial roots

$f(x) = (x-2)^2 - 1$



$x = 1, 3$ are roots of the quadratic.

Relationship Nonlinear Eqs

$F(\underline{x}) = 0$

$\Downarrow$

minimize $\|F(x)\|_2^2$

Doesn't quite work

s Between

+ Optimization

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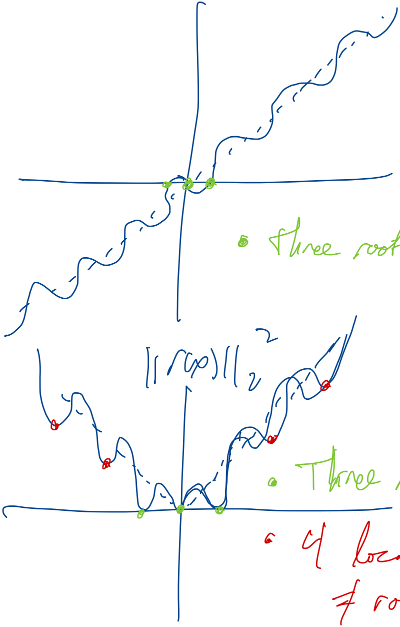
minimize $f(\underline{x})$

$$\Downarrow$$

$g(\underline{x}) = 0$

Non-linear Eq!

$r(x) = x - \sin 5x = 0$



- Three roots

- Three roots
- 4 local mins $\neq$ roots.

How to design algs for $F(\underline{x}) = 0$? $F : \mathbb{R}^n \to \mathbb{R}^m$

Use Taylor's theorem

↳ Newton's method.

Newton's Method for Non-linear Eqs.

$F(\underline{x}) \longrightarrow J(\underline{x})$ the Jacobian

$J(x) : m \times n$

$$J_{ij} = \begin{array}{c} 1 \\ \vdots \\ m \end{array} \overbrace{\frac{\partial F_i}{\partial x_j}}^{1 \text{ to } n}$$

"Transpose" of what we called the gradient

$f(\underline{x}) \longrightarrow g(\underline{x}) \in \mathbb{R}^n$

$J(\underline{x}) \longrightarrow g(x)^T \in \mathbb{R}^{1 \times n}$.

*Rows of the Jacobian" are gradients of the functions in F.

Newton's Method

x_0 - given

~~$x_{k+1} = x_k - J(x_k)^{-1} F(x_k)$~~

$J(x_k)\, p_k = -F(x_k)$

$x_{k+1} = x_k + \alpha_k p_k$.

How to pick α?

↳ Opt.: use $f(x)$ to find "smaller"

↳ Non-lin eq...?

Merit Functions

Help guide us to better points.

$\|F(x)\|_2^2$ → One Example.

More Examples in book.

⊛ Idea is guidance about how to proceed.
Not about min $\|F(x)\|_2^2$.

What if J is singular?

→ Exact dir isn't so impor.

find p_k s.t.

$$\|J p_k + F_k\|_2^2 \le \eta \|F_k\|$$

What if no Jacobian?

- Quasi-Newton Approx to J.
- Krylov-Newton method!

We need J to get p in

$$Jp = -F.$$

A Krylov method solves a system $A\underline{z} = \underline{b}$ and only uses matrix vector products w/ A.

Given an alg for arb. y.

$$A \cdot y \leftarrow \text{(matvec)}$$

Then a Krylov method use this alg repeatedly to solve

$$A z = \underline{b} .$$

This is awesome because

$$J \cdot y \quad \text{(matvec)}$$

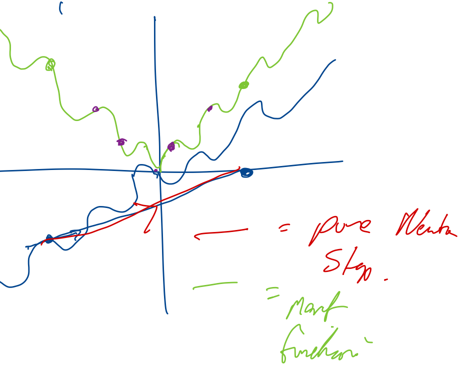
$$\simeq \frac{F(x_k + h y) - F(x_k)}{h}$$

Use $h = \sqrt{\varepsilon}$ to get a matvec!

Use ideas from class + apply to non-linear Eqns!

Ive focused on the most "new" material.

Intuition behind Merit function



Homotopy methods

- Useful when $J(\underline{x}) = 0$.

Take $F(\underline{x})$, create a new function

$$H(x, \lambda) = (1-\lambda)(\underline{x}-\underline{a}) + \lambda F(\underline{x}).$$

If $\lambda = 0$, what are the roots of $H(x, \lambda)$?

Quiz!

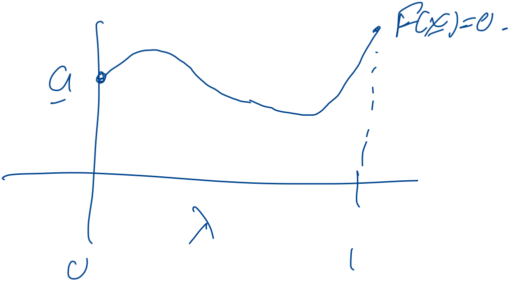
If $\lambda = 0$, Then

$$H(\underline{x};0) = (\underline{x}-\underline{a}) = 0$$

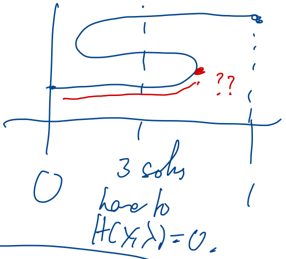
$$\Leftrightarrow \underline{x} = \underline{a}.$$

@ $\lambda = 0$, we know answer.

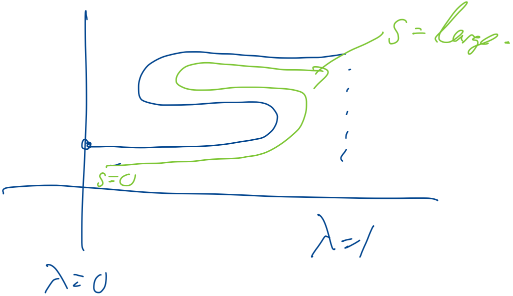
Idea: Track the path



But, we can get multiple roots.



What to do?



Idea: follow the curve, not λ.

We parameterize $x(s), \lambda(s)$ s.t.

$$H(x(s), \lambda(s)) = 0.$$

for "most" F, (see book), this is continuous.

find a path

If $\lambda(s^*) = 1$,
done!

We can use an

the path.

$$\frac{dx}{ds} = \boxed{\text{See book}}$$

$$\frac{d\lambda}{ds} = \boxed{\text{See book}}$$

$$x(0) = \underline{a}, \quad \lambda(0) =$$

$x(s)$, $\lambda(s)$

then we are

ODE to follow

too long to work

0. Use ODE Solver.

Theorem 11.9 in NW.

This works!

if $F(\underline{x})$ is twice cont-diff.

then for all $\underline{a}$ but a set of measure 0, then the update is well-defined

The path converges/diverges in finite time.