OPTIMALITY AND DUALITY FOR LINEAR PROGRAMS

David F. Gleich

February 8, 2023

Recall the standard form for a linear program:

The material here is from Chapter 13 in Nocedal and Wright, but some of the geometry comes from Griva, Sofer, and Nash.

$$\begin{array}{ll} \text{minimize} & \mathbf{c}^T\mathbf{x} \\ \text{subject to} & A\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \ge 0. \end{array}$$

1 THE DUAL OF AN LP

We'll revisit the dual later. I won't have enough time to talk about it in this lecture. But for reference, my favorite way of defining the dual is via the "simple LP"

$$\begin{array}{ll} \underset{\mathbf{x}}{\text{minimize}} & \mathbf{c}^T\mathbf{x} \\ \text{subject to} & A\mathbf{x} \le \mathbf{b}. \end{array}$$

It's Lagrangian is:

$$\mathcal{L}(\mathbf{x}, \boldsymbol{\lambda}) = \mathbf{c}^T\mathbf{x} - \boldsymbol{\lambda}^T(\mathbf{b} - A\mathbf{x}),$$

which can be "tranposed" to yield:

$$\mathcal{L}(\mathbf{x}, \boldsymbol{\lambda}) = \mathbf{x}^T\mathbf{c} + \mathbf{x}^T A^T\boldsymbol{\lambda} - \mathbf{b}^T\boldsymbol{\lambda} = (-\mathbf{b})^T\boldsymbol{\lambda} - \mathbf{x}^T(A^T\boldsymbol{\lambda} - \mathbf{c})$$

which is the Lagrangian of

$$\begin{array}{ll} \underset{\boldsymbol{\lambda}}{\text{minimize}} & -\mathbf{b}^T\boldsymbol{\lambda} \\ \text{subject to} & \mathbf{c} \le A^T\boldsymbol{\lambda}, \boldsymbol{\lambda} \ge 0 \end{array}$$

where $\mathbf{x}$ are now the Lagrange multipliers.

The Lagrange multipliers λ are often called *dual variables* for this reason.

2 KKT CONDITIONS ARE NECESSARY AND SUFFICIENT

Let $\boldsymbol{\lambda}$ be the Lagrange multipliers for the equality constraints and $\mathbf{s}$ be the multipliers for the inequality constraints. Then the KKT conditions for the primal LP are:

$$\begin{aligned} A^T\boldsymbol{\lambda} + \mathbf{s} &= \mathbf{c} \\ A\mathbf{x} &= \mathbf{b} \\ \mathbf{x} &\ge 0 \\ \mathbf{s} &\ge 0 \\ \mathbf{x}^T\mathbf{s} &= 0. \end{aligned}$$

For the rest of the course, you might want to commit these conditions to memory! They'll be very important.

In general, the KKT conditions are only the necessary conditions for a local minimum. However, for an LP, we'll show that they are also sufficient. In other words, any point $\mathbf{x}$ that satifies these conditions is a solution, that is, a local minimizer and a global minimizer.

First, note that for any solution $(\mathbf{x}^*, \boldsymbol{\lambda}^*, \mathbf{s}^*)$ that satifies the KKT conditions, we have that

$$\mathbf{c}^T\mathbf{x}^* = \mathbf{b}^T\boldsymbol{\lambda}^*.$$

Quiz Show this using the KKT conditions!

Now, consider any other feasible point $\mathbf{f}$ where $\boldsymbol{A}\mathbf{f} = \mathbf{b}$, $\mathbf{f} \geq 0$. We can show that $\mathbf{c}^T\mathbf{f} \geq \mathbf{c}^T\mathbf{x}^*$ directly:

$$\mathbf{c}^T\mathbf{f} = (\boldsymbol{A}^T\boldsymbol{\lambda}^* + \mathbf{s}^*)^T\mathbf{f} = \mathbf{b}^T\boldsymbol{\lambda}^* + \mathbf{f}^T\mathbf{s}^* \geq \mathbf{b}^T\boldsymbol{\lambda}^* = \mathbf{c}^T\mathbf{x}^*$$

because $\mathbf{f} \geq 0$ and $\mathbf{s}^* \geq 0$.