Recall the standard form for a linear program:

\[
\begin{align*}
\text{minimize} & \quad c^T x \\
\text{subject to} & \quad Ax = b \\
& \quad x \geq 0.
\end{align*}
\]

1 THE DUAL OF AN LP

We'll revisit the dual later. I won't have enough time to talk about it in this lecture. But for reference, my favorite way of defining the dual is via the "simple LP"

\[
\begin{align*}
\text{minimize} & \quad x^T c \\
\text{subject to} & \quad A x \leq b.
\end{align*}
\]

It's Lagrangian is:

\[\mathcal{L}(x, \lambda) = c^T x - \lambda^T (b - Ax),\]

which can be “tranposed” to yield:

\[\mathcal{L}(x, \lambda) = x^T c + x^T A^T \lambda - b^T \lambda = (-b)^T \lambda - x^T (A^T \lambda - c)\]

which is the Lagrangian of

\[
\begin{align*}
\text{minimize} & \quad -b^T \lambda \\
\text{subject to} & \quad c \leq A^T \lambda, \lambda \geq 0
\end{align*}
\]

where \(x\) are now the Lagrange multipliers.

The Lagrange multipliers \(\lambda\) are often called dual variables for this reason.

2 KKT CONDITIONS ARE NECESSARY AND SUFFICIENT

Let \(\lambda\) be the Lagrange multipliers for the equality constraints and \(s\) be the multipliers for the inequality constraints. Then the KKT conditions for the primal LP are:

\[
\begin{align*}
A^T \lambda + s &= c \\
A x &= b \\
x &\geq 0 \\
s &\geq 0 \\
x^T s &= 0.
\end{align*}
\]

For the rest of the course, you might want to commit these conditions to memory! They'll be very important.

In general, the KKT conditions are only the necessary conditions for a local minimum. However, for an LP, we'll show that they are also sufficient. In other words, any point \(x\) that satisfies these conditions is a solution, that is, a local minimizer and a global minimizer.

First, note that for any solution \((x^*, \lambda^*, s^*)\) that satisfies the KKT conditions, we have that

\[c^T x^* = b^T \lambda^*.
\]

Quiz Show this using the KKT conditions!
Now, consider any other feasible point $f$ where $Af = b, f \geq 0$. We can show that $c^Tf \geq c^Tx^*$ directly:

$$c^Tf = (A^T\lambda^* + s^*)^Tf = b^T\lambda^* + f^Ts^* \geq b^T\lambda^* = c^Tx^*$$

because $f \geq 0$ and $s^* \geq 0$. 