Lecture 9

HW2 due in a week!

- How to pick the ω in $M\omega$?
  - If Jacobi to get the ω right.
- Why does G-S converge faster than Jacobi?
- How to generate ω so that the ω is n^{-1}? ✓
- Jacobi over-GS? ✓

$N \cap D$ vs. $M_N \cap N$ in Jacobi?

$\rho(A)$ vs. $\kappa(A)$

P vs. M? ✓

Does choice of P impact convergence?

Convergence proof.

How to interpret the decay chart? Convergence criterion for Jacobi?

Good discussion on Gauss-Seidel?

How results vs M^{-1} can give have $\rho(D_i^{-1}M_i) < 1$ for some of the not others?

- Yes, it's possible to pick M / P so that $A\vec{x} = \vec{b}$ does or does not converge.
- M vs. P.

M - pick what variable to update for Gauss Seidel.

P - parallelizable matrix so after permute. Varied i in eg. i.

What results converge $\rho(D^{-1}N)$

This is small if A is "locally diagonally dominant" so pivots P/M so we use this larger Estimate

$\Rightarrow$ Mono weight matrix between inverse of eg. vals.

Why GS faster than Jacobi?

$\rightarrow$ always true?

Theorem:

$\rho(M_{GS}^{-1}M_{GS}) \leq \rho(M_J^{-1}M_J)$

Doubt theorem:

Jacobi: $2T$ iterations.

GS: 300 iterations

$A = X J X^{-1}$

$J = \begin{bmatrix} J_1 & & \\ & J_2 & \\ & & \ddots \\ & & & J_k \end{bmatrix}$

$J_i = \begin{bmatrix} \lambda_i & 1 & 0 \\ & \ddots & \\ 0 & & \lambda_i \end{bmatrix}$ σ Gauss Seidel

or

$J_i = \begin{bmatrix} \lambda_1 & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda_i \end{bmatrix}_n$

n

If

$A = \begin{bmatrix} I & B \\ B^T & I \end{bmatrix}$

Then Gauss-Seidel $\Leftrightarrow$ 2 Steps of Jacobi.

Jacobi

$D^{-1}N = \begin{bmatrix} 0 & B \\ B^T & 0 \end{bmatrix}$ 2 steps $\Rightarrow \begin{bmatrix} BB^T & 0 \\ 0 & B^TB \end{bmatrix}$

Gauss Seidel

$N^{-1}N \quad \begin{bmatrix} I & 0 \\ B^T & I \end{bmatrix}^{-1} \begin{bmatrix} 0 & B \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & B \\ 0 & B^TB \end{bmatrix}$

Gauss Seidel $\Leftrightarrow$ Good choice of A has I on the diagonal, as $A = S_{ym}$ pos def.

Jacobi + Ring theorem

$A = D + N$

If A is diag. dom, then

$D^{-1}N$ is "like" a contraction

A contract. Map

$M(x)$ means.

$\|M(x)\| \leq \|x\|$.

$\rho(A) = \max |\lambda_i|$

$\kappa(A) = \|A\|_2 \cdot \|A^{-1}\|_2$

$= \frac{\sigma_{max}}{\sigma_{min}}$ sing. value

If $A = A^T$, then

$\sigma_i = |\lambda_i|$

$\Rightarrow \kappa(A) = \frac{|\lambda_{max}|}{|\lambda_{min}|}$

$A\vec{x} = \vec{b}$

$\downarrow$

$(M + N)\vec{x} = \vec{b}$

$M\vec{x} = \vec{b} - N\vec{x}$

$\boxed{\vec{x} = M^{-1}(\vec{b} - N\vec{x})}$

$\downarrow$

$\vec{x}_{k+1} = M^{-1}(\vec{b} - N\vec{x}_k)$ Then iterate (Jacobi).

Jacobi $M = D$, $N = (A - D)$

$\vec{x} - \vec{x}_{k+1} = M^{-1}\vec{b} - M^{-1}N\vec{x} = M^{-1}N(\vec{x} - \vec{x}_k)$

$-M^{-1}\vec{b} + M^{-1}N\vec{x}_k$

$\vec{e}_{k+1} = -M^{-1}N\vec{e}_k$

$\vec{e}_{k+1} = (-M^{-1}N)^k\vec{e}_0$

To get $\|\vec{e}_k\| \to 0$

$\Leftrightarrow$

$\rho(M^{-1}N) < 1$.

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