Lecture 25

Q: Take any set of input data $x_1, \ldots, x_n$.

If $alg(x)$ is any algorithm that uses only addition, subtraction, multiplication, then all outputs are $poly(x_1, \ldots, x_n)$

- What polynomial is what measure is CG implicitly using?
- Do polys come up in iterative method besides CG?
- Where is the gradient in conjugate gradient? ✓
- polys orthogonal over all intervals? ✓

- Can we store Q in GMRES implicitly to save space? ✓
- When does GMRES finish in at most n steps ($n \times n$ matrix)
- What's the role of the minimal poly in the terminal behavior. ✓
- Where else do orthog. polys come up? ✓

- Can we predict what k gives a good bound on $\|r_k\|$? ✓
- When/why does the GMRES code run until max iter? ✓
- Why did we do the Padé approx to $\cos(x)$ for x after scaling to $\|2^{-m} A\|_\infty \leq 1$. ✓

- Predict k st $\|r_k\| \leq \tau$?

for CG…

$$\|x - x_k\|_A \leq 2\|x - x_0\|_A \left(\frac{\sqrt{\kappa(A)} - 1}{\sqrt{\kappa(A)} + 1} \right)^k .$$

(for some norm.)

In GMRES (i.e. Arnoldi)

- Need Q_k explicitly.
- Want re-orth if you want converge in n-steps.

CG/MinRes (i.e. Lanczos)

- Only need 2 terms of V_k.

In practice

Run GMRES for m steps.

get x_m, then solve $A(y + x_m) = b$

Use GMRES for m steps.

⋮

Keep repeating until you are happy!

$Ax = b$ (GMRES)

$\min \|Ax - b\|$ (LSQR)

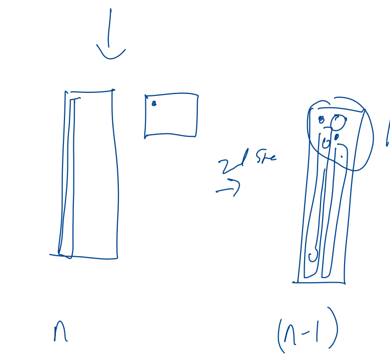
↓

$$\begin{bmatrix} I & A \\ A^T & 0 \end{bmatrix} \begin{bmatrix} r \\ x \end{bmatrix} = \begin{bmatrix} b \\ 0 \end{bmatrix}$$

To think about:

If Q is an orthog. matrix, what happens if you run QR?

↓



$\min \frac{1}{2} x^T A x - x^T b$

Then $g(x) = Ax - b$
= residual

$\int_a^b f(x) g(x)\, dx = 0$

Can we have

$\int_a^b f(x) g(x)\, dx = 0 \Rightarrow F(b) - F(a) = 0$ for all a,b.

for all a, b?.

good question!

Do f, g actually exist?

Good Exercise.

MinRes

$\min \|Ax_k - b\|$
$x_k \in \mathcal{K}_k(A, b)$

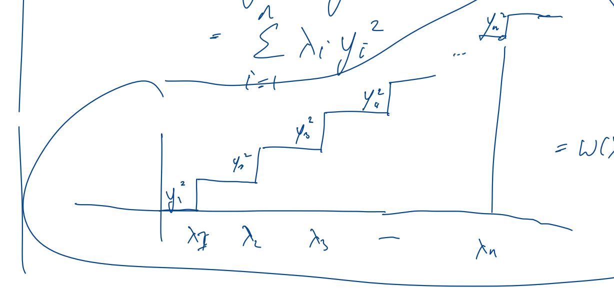
⇕

$\min \|AV_k y_k - b\|$

Show $r_k^T A r_j = 0$ for all steps of MinRes.

The orthogonal polynomial is a matrix!

$$x^T A x = x^T V \Lambda V^T x = y^T \Lambda y = \sum_{i=1}^{n} \lambda_i y_i^2$$



In CG the weights are related to the function

$$\int_a^b f(x) g(x)\, dw(x) = \sum_{i=1}^{n} f(\lambda_i) g(\lambda_i) w_i$$