Lecture 24

Q: How can we get $\|r_k\|_2$ from CG w/o computing it if we use the Lanczos CG.

- Google says CG isn't backward stable, why don't we care about this?
- Why doesn't "breakdown" in Lanczos (due to FP) not cause a problem for CG?
- CG vs. Steepest Descent. (3x!)
- Does Lanczos need pos. def? Where does CG use this?
- How does T_k "approximate" A?

- Why is CG called CG?
- Is cholesky on tridiag. still $O(n)$?

$$\boxed{A V_k = V_{k+1} \underbrace{T_{k+1}}}$$

$$= \begin{bmatrix} T_k \\ \beta_{k+1} e_k^T \end{bmatrix}$$

for CG $\Leftrightarrow$

find y st

$$\min \tfrac{1}{2} y^T T_k y - \|b\| e_1^T y$$

$$\Leftrightarrow T_k y = \|b\| e_1$$

So

$$\|b - A V_k y\| =$$

$$\| \|b\| V_{k+1} e_1 - V_{k+1} T_{k+1} y \|$$

$$\boxed{\| \|b\| e_1 - T_{k+1} y \|}$$

for CG

$$= \left\| \boxed{\begin{bmatrix} \|b\| \\ 0 \\ \vdots \\ 0 \end{bmatrix} - \begin{bmatrix} T_k \\ 0 \; \beta_k \end{bmatrix} y} \right\| = 0 \text{ for CG.}$$

$|\beta_k y_k|$.

$\boxed{\text{This relationship holds!}}$

Even if V isn't orthogonal..

Lanczos doesn't need pos. def!

$A V_k = V_{k+1} T_{k+1}$ is good for any symmetric A.

$$A V_k = V_{k+1} T_{k+1}$$

$$V_k^T A V_k = \underbrace{V_k^T V_{k+1}} T_{k+1}$$

$$[I \; 0] \begin{bmatrix} T_k \\ \beta_{k+1} e_k^T \end{bmatrix}$$

$$= T_k.$$

SD

$$\min_\alpha \tfrac{1}{2} x^T A x - x^T b$$

st $x_{k+1} = x_k + \alpha \underbrace{(b - A x_k)}_{-g_k}$

$x_0 = 0.$

CG

$$\min \tfrac{1}{2} x^T A x - x^T b$$

st. $x_k \in \mathcal{K}_k(A, b)$

$= \operatorname{span}(b, Ab, \ldots, A^k b)$

$= V_k y$

Note $\mathcal{K}_k(A, b)$ contains

$x_k, \; b - A x_k$.

If you want to start CG at a vector s then solve

$$A(z + s) = b$$

or

$$\Rightarrow$$

$$\boxed{Az = b - As.}$$

and let $z_0 = 0.$

BW Stable

f = math function

alg = computer implementation

BW stable if

$$\text{alg}(x) = f(x + \delta)$$

for "small" δ.

f for CG is

$\boxed{\text{find } x_k \text{ st } \|b - A x_k\| \le \tau}$.