Lecture 23

Question:

In $\boxed{AV_k = V_{k+1} T_{k+1}}$

with $V_k^T V_k = I$.

Are there any degrees of freedom?

$V_k e_1 = b/\|b\|$

1. Are there cases when

$[b, Ab, \ldots, A^k b]$ ✓

doesn't have dimension $k+1$

- For CG do we need to run Arnoldi to completion or do we get a good answer part way through? (2 variations)
- Can we apply Lanczos to non-sym matrices? Yes – but Bi-Lanczos and it doesn't work well.
- Why does projection onto the Krylov subspace help solve $Ax = b$ / Eigenvalue ✓

→ Why is this a good subspace

→ Why faster than stationary?

How do rounding Errors impact orthogonality of Lanczos.
↳ Can we mitigate? ✓

Lanczos → Lanczos-tri.

How does Arnoldi build an orthogonal basis for $K_k(A,b)$? ✓

Why does Arnoldi rely on a Hessenberg matrix? ✓

Can we say projection to V_k reveals the Tridiag Structure? That's why it works?

Krylov picks a new polynomial @ Each step. Does this impact Convergence?

it choose the "best" poly at Each iteration?

$$A[\underline{v}_1\ \underline{v}_2\ \underline{v}_3] = [\underline{v}_1$$

$\underline{v}_1$ → input / fixed.

$$A\underline{v}_1 = \alpha_1 \underline{v}_1 + \beta_2 \underline{v}_2$$

$$\alpha_1 = \underline{v}_1^T A \underline{v}_1 \rightarrow$$

$$\beta_2 \underline{v}_2 = A\underline{v}_1 - \alpha_1 \underline{v}_1$$

β_2, $\underline{v}_2$ are "forced

$$\underline{v}_2\ \underline{v}_3\ \underline{v}_4] \begin{bmatrix} \alpha_1 & \overset{\gamma_2 ?}{\boxed{\beta_2}} & \gamma_3 & \\ \beta_2 & \alpha_2 & \boxed{\beta_3} & \\ & \beta_3 & \alpha_3 & \\ & & \beta_4 & \end{bmatrix}$$

v_1 - fixed

α_1 - fixed

β_2 - fixed up to sign

v_2 - fixed up to sign.

$$A\underline{v}_2 = \beta_2 \underline{v}_1 + \alpha_2 \underline{v}_2 + \beta_3 \underline{v}_3]$$

$$y = A v_2 = \underline{\gamma_2 v_1 + \alpha_2 v_2 + \beta_3 v_3}]$$

$$\boxed{\begin{gathered} \underbrace{v_1^T y} = \gamma_2 v_1^T v_1 = \gamma_2 \\ \underline{v}_1^T A \underline{v}_2 = v_2^T A v_1 \text{ by symmetry} \end{gathered}}$$

$$\boxed{v_2^T v_1 = 0.}$$

fixed!

also $\|\underline{v}_2\| = 1.$

: up to sign.

$$v_2^T A v_2 = \alpha_2 \to \text{fixed?}$$

$$\beta_3 = \|y - \gamma_2 v_1 - \alpha_2 y\| . \text{ up to sign.}$$

What happens if you Relax orthogonality?

$AV_k = V_{k+1} T_{k+1}$ but we don't assume orthogonality?

Suppose V_k wasn't orthogonal! (But

Then we choose α_k s.t.:

$$\| \overbrace{A v_k - \beta_k v_{k-1}}^{\text{"b"}} - \overbrace{\alpha}^{\text{"x"}} \overbrace{v_k}^{\text{"A"}} \|_2 =$$

is as small as possible.

Then we will get orthog V_k and β_k will be as small as poss

$\|v_i\| = 1$)

β_{k+1}

$$A\underline{x} = \underline{b}$$

$$\Downarrow$$

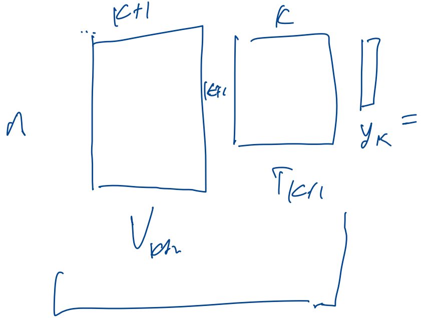
If $x_k \in \mathcal{K}_k(A, b)$

Then $\underline{x}_k = V_k y_k$ for some y_k.

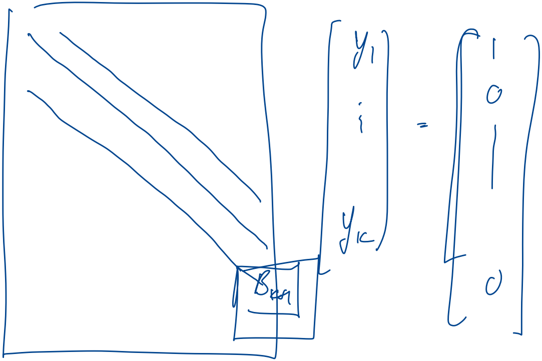
$$\underbrace{A V_k}_{\downarrow} y_k = b$$

$$V_{k+1} T_{k+1} y_k = V_{k+1} \underline{e}_1 \cdot \|b\|$$

To



$$\boxed{\text{If } T_{k+1} y_k = \underline{e}_1}$$



I'm done! ?

CG:

$$\min \frac{1}{2} x^T A x - x^T b$$

$$\text{s.t.} \quad x_k \in \mathcal{K}_k(A, b)$$

MinRes

$$\min \|Ax - b\|_2$$

$$\text{s.t.} \quad x_k \in \mathcal{K}_k(A, b)$$

if you stop iteration when $\boxed{\frac{\|Ax - b\|_2}{\|b\|_2} \le \text{tol.}}$

then MinRes must use no more iterations than CG..

Why faster than stationary?

SD computes

$x_k \in \mathcal{K}_k(A, \underline{b})$.

A

$[\underline{b}, A\underline{b}, \ldots A^k \underline{b}] = Q$

I want to update to

$[b, Ab, \ldots A^k b, A^{k+1} \underline{b}$

$M_{k+1} = [QR \;\; \underline{v}]$

$Q^T M_{k+1} = [R \;\; Q^T v] =$

$R = M_k$

$\} = M_{k+1}$

$\boxed{A Q_k} = Q_{k+1} H_{k+1}$

Assume Q_k spans $[b, Ab, \ldots A^{k-1} b]$

Then $A Q_k$ has vectors in

$[Ab, \ldots, A^k b]$.