Lecture 9

HW4 Posted

Today: Jacobi and Gauss Seidel methods for $Ax=b$

--> "Equation-wise" methods

And how Gauss-Seidel is sometimes equivalent to Coordinate Descent!

(Probably next time.)

Next up: "Simple algorithms for Eigenvalues (Power Method)

Then we are done with Simple methods and on to finite termination & equation elimination methods

Solving $A\underline{x} = \underline{b}$

Let $A = \begin{bmatrix} a_1^T \\ a_2^T \\ \vdots \\ a_n^T \end{bmatrix}$

Then $A\underline{x} = \underline{b} \iff$

$a_1^T \underline{x} = b_1$

$a_2^T \underline{x} = b_2$

$\vdots$

$a_n^T \underline{x} = b_n$

Look back at Eq 27

$a_{27}^T \underline{x} = b_{27}$

$\sum_k A_{27,k} x_k = b_{27}$

Suppose we have a guess $\underline{x}$

Let change x_{27} to make Eq 27 hold!

Solve:

$\sum_{k \neq 21} A_{27,k} x_k^{(\text{old})} + A_{27,21} x_{21} = b_{27}$

$\Rightarrow$

$x_{21} = \dfrac{\left(b_{27} - \sum_{k \neq 21} A_{27,k} x_k^{(\text{old})}\right)}{A_{27,21}}$

We need x_{21} to start in Eq 27 $\iff$ $A_{27,21} \neq 0$.

Variable

Good. Use all rows to update all variables! Get first! A quiz!

Example:

$A = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} \quad b = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

What is $\underline{x}$? (Your quiz!)

Solve: there is a trick!

$x = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$A = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$

Let's Solve this using row 2 to update x_2

$x_1^{(\text{new})} = \dfrac{1}{A_{21}}\left(b_2 - A_{22} x_2^{(\text{old})}\right)$

$= \dfrac{1}{-2}\left(1 - x_2^{(\text{old})}\right)$

$x_2^{(\text{new})} = \dfrac{1}{A_{22}}\left(b_1 - A_{11} x_1^{(\text{old})}\right)$

$= \dfrac{1}{2}\left(2 - x^{(\text{old})}\right)$

$x^{(\text{old})} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \longrightarrow$

$x^{(\text{new})} = \begin{bmatrix} -1/2 \\ 1/2 \end{bmatrix}$

(Check this!)

In general, we read the row M to tell us what row to use to update variable.

$M(i,j) = 1$ if row j is used to update x_i

0 otherwise.

Also $(M^{-1})_{ij}$ if row j used to update x_i.

We also read $A_{ji} \neq 0$ when $M_{ij} = 1$.

We need some tools still. If we use row j to update x_i ($A_{ji} \neq 0$)

And we can update x_i from any other row.

$\sum_j M(i,j) = 1 \quad \forall i$

$\sum_i M(i,j) = 1 \quad \forall j$

(M is a permutation matrix)

Does M always exist?

An Easy Case

$A_{ii} \neq 0 \Rightarrow M_{ii} = 1$

$\Rightarrow$ Row is in result!

$x_1^{(\text{new})} = \dfrac{1}{A_{M_{1,1}}}\left(b_{M_1} - \sum_{k \neq M_1} A_{M_1,k} x_k^{(\text{old})}\right)$

$\vdots$

$x_i^{(\text{new})} = \dfrac{1}{A_{M_{i,i}}}\left(b_{M_i} - \sum_{k \neq M_i} A_{M_i,k} x_k^{(\text{old})}\right)$

$x^{(\text{new})} = \underbrace{\begin{bmatrix} 0 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & \cdots & 0 \end{bmatrix}}_{U_M} \left(b_M - N_M \underline{x}^{(\text{old})}\right)$

$(U_M)_{ij} = \dfrac{1}{A_{M_i,i}}$

$(b_M)_i = b_{M_i}$

$(N_M)_{ij} = \begin{cases} A_{M_i,j} & \text{if } j \neq M_i \\ 0 & \text{if } j = M_i \end{cases}$

$\sum_k (U_M)_{ik} x_k^{(\text{old})} = \sum_{k \neq M_i} A_{M_i,k} x_k^{(\text{old})}$

Hurray! Matrix Permutation!

$A\underline{x} = \underline{b} \iff$

$PA\underline{x} = P\underline{b}$ for some permutation matrix P.

If MA P (or $M^T A P$)

The "Trick" that in most books doesn't mention P or M.

They will use $M = I$ (and assume you know what P)

You don't need M if you permute the system & P.

Using $M \iff$

Solve $M^{-1}Ix$ and look @ $PA\underline{x} = P\underline{b}$

The gold case we have

$x^{(\text{new})} = \dfrac{1}{A_{ii}}\left(b_i - N_{ii} x^{(\text{old})}\right)$

Does this converge?

$x^{(\text{new})} = U_{ii}\left(b_i - N_{ii} x^{(\text{old})}\right)$

If x^* is a soln, then

$x^* = U_{ii}\left(b_i - N_{ii} x^*\right)$

$x^{(k)} = x^* =$

$U_{ii}(b_i - N_{ii} x^{(k-1)}) -$

$U_{ii}(b_i - N_{ii} x^*) =$

$-U_{ii} N_{ii} (x^{(k-1)} - x^*) =$

$(-U_{ii} N_{ii})^k (x^{(0)} - x^*)$

Start.

So if $(-U_{ii} N_{ii})^k \to 0$

The converges.

If $\rho(U_{ii} N_{ii}) < 1$

It will converge for any RHS.

Gauss-Seidel is a big m

Trick:

$x_1^{(\text{new})} = \dfrac{1}{A_{M_{1,1}}}\left(b_{M_1} - \sum_{j \neq 1} A_{M_1,j} x_j^{(\text{old})}\right)$

$x_2^{(\text{new})} = \dfrac{1}{A_{M_{2,2}}}\left(b_{M_2} - \sum_{j \neq 2} A_{M_2,j} x_j^{(\text{old})}\right)$

$\cancel{- A_{M_2,1} x_1^{(\text{old})}}$

$- \sum_{j > 2} A_{M_2,j} x_j^{(\text{old})}$

To finish on Thursday.