Lecture 23

We continue with
Krylov subspace methods

- The result of the Arnoldi process spans the Krylov space

- The Lanczos process specializes the Arnoldi process on symmetric matrices and gives a Tridiagonal reduction.

- Krylov methods give very simple iterative methods at a high level because the residual can be expressed very simply.

- The CG method arises through one such transformation.

Alg for Arnoldi

Input Square A, $\underline{b}$

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Let $\underline{v}_1 = \underline{b}/\|\underline{b}\|$
for i = 1 to k
    $\underline{y} = A\underline{v}_i$
    for j = 1 to k
        $H_{j,i} = \underline{v}_j^T \underline{y}$
        $\underline{y} = \underline{y} - H_{j,i}\underline{v}_j$
    End
    $H_{i+1,i} = \|\underline{y}\|$
    $\underline{v}_{i+1} = \underline{y}/H_{i+1,i}$
End.
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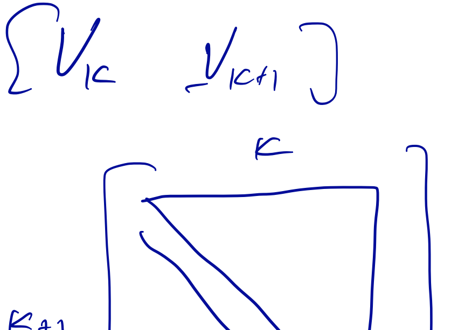
$A V_k = V_{k+1} H_{k+1}$

V_k = orthog n by k

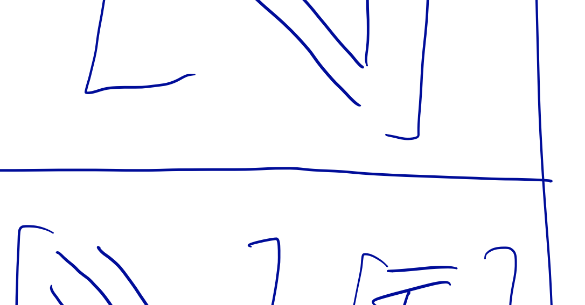
$V_{k+1} = [V_k \ \ \underline{v}_{k+1}]$

$H_{k+1} =$



Upper Hessenberg" $k+1$, k

$T_{k+1} =$



$= \begin{bmatrix} T_k \\ \rho e_k^T \end{bmatrix}$

ρ

Result 1 = Arnoldi on a symmetric matrix $A = A^T$ gives a tri-diag "H_k"

Recall: Arnoldi

$$\underset{n}{A}\ \underset{k}{V_k} = n[\overset{k}{V_k}\ \overset{1}{\underline{u}_{k+1}}]\begin{bmatrix} \overset{k}{\bar{H}_k} \\ \rho_{k+1}e_k^T \end{bmatrix}\begin{matrix} k \\ 1 \end{matrix}$$

$$V_k^T A V_k = V_k^T [V_k \ \ \underline{u}_{k+1}]\begin{bmatrix} \bar{H}_k \\ \rho_{k+1}e_k^T \end{bmatrix}$$

$$= [\underbrace{V_k^T V_k}_{=I} \ \ \underbrace{V_k^T \underline{u}_{k+1}}_{=0}]\begin{bmatrix} \bar{H}_k \\ \rho_{k+1}e_k^T \end{bmatrix}$$

$$= \bar{H}_k$$

$V_k^T A V_k = \bar{H}_k$

$(V_k^T A V_k)^T = V_k^T A^T V_k$

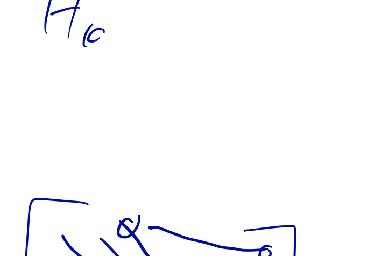
$= \bar{H}_k^T$

but $A = A^T$, so

$\bar{H}_k = \bar{H}_k^T$

$\Leftrightarrow$

$\bar{H}_k =$



$\Rightarrow \bar{H}_k$ is tridiag. $\blacksquare$

Result 2: The matrix
V_k is an orthog basis for
span $(\underline{b}, A\underline{b}, \ldots, A^{k-1}\underline{b})$

Pf: by induction

$k = 1$

$\underline{v}_1 = \underline{b}/\|\underline{b}\| \Leftrightarrow$ span $(\underline{b})$.

Let's assume
V_k is a basis for
span $(\underline{b}, A\underline{b}, \ldots, A^{k-1}\underline{b})$

$V_k \underline{y} = X\underline{c} \qquad X = [\underline{b}, A\underline{b}, \ldots A^{k-1}\underline{b}]$

Need to show
$V_{k+1} = [V_k \ \underline{u}_{k+1}]$
spans $(\underline{b}, A\underline{b}, \ldots A^k\underline{b})$

Claim: we just need to show that

$(A^k\underline{b})^T \underline{u}_{k+1} \neq 0$.

Why?
Consider

$A^k\underline{b} = V_{k+1}\underline{c}$ ← need to show $\underline{c}$ exists

We will show
$\underline{u}_{k+1} = \gamma A^k\underline{b}$ + stuff in $\underline{b}, \ldots, A^{k-1}\underline{b}$

Need to use def of $\underline{u}_{k+1}$

$$\underline{u}_{k+1} = \rho\Big[\underline{y} - \sum_{j=1}^{k}(\underline{u}_j^T\underline{y})\underline{u}_j\Big]$$

$\underline{y} = A\underline{u}_k$

$$\rho = \frac{1}{\|\underline{y} - \sum_{j=1}^{k}(\underline{u}_j^T\underline{y})\underline{u}_j\|}$$

By assumpt $\underline{u}_j = X\underline{c}_j$

$X = [\underline{b}, \ldots A^{k-1}\underline{b}]$

$\underline{y} = AX\underline{c}_k$
$= [A\underline{b}, \ldots A^k\underline{b}]\underline{c}_k$

$$\underline{y} = \sum_{i=1}^{k} A^i\underline{b}\,[\underline{c}_k]_i$$

We need $[\underline{c}_k]_k \neq 0$.

We can assume
$[\underline{c}_k]_k \neq 0$ because otherwise
$\underline{u}_k \in$ Span$(\underline{b}, \ldots A^{k-2}\underline{b})$
$\Rightarrow$ Arnoldi done!

$$\sum_{j=1}^{k}\underline{u}_j^T\underline{y}\,\underline{u}_j =$$

$$\sum_{j=1}^{k}\underbrace{(\underline{c}_j^T X^T A X\underline{c}_j)}_{\text{Constant} = \theta_j}X\underline{c}_j$$

$$= \sum_{j=1}^{k}\theta_j X\underline{c}_j$$

$\Leftrightarrow$

all powers of $(\underline{b}, \ldots, A^{k-1}\underline{b})$.

So we have

$\underline{u}_{k+1} = \rho A^k\underline{b}[\underline{c}_k]_k$ + things in $\underline{b}, \ldots A^{k-1}\underline{b}$ $\blacksquare$ ☺

Remember our goal

We want to find

$A\underline{x} \simeq \underline{b}$

in the Krylov Subspace.

$\underline{x}_k \in$ Span$(\underline{b}, \ldots, A^{k-1}\underline{b})$

$\Leftrightarrow$

$\underline{x}_k = V_{k+1}\underline{y}$

A very simple idea

Pick $\underline{y}$ to minimize

$\|\underline{b} - A\underline{x}_k\|_2$

$\underline{x}_k = V_k\underline{y}$

Not Consistent

$\Leftrightarrow$

$\|\underline{b} - AV_k\underline{y}\|$

$AV_k = V_{k+1}H_{k+1}$

$$H_{k+1} = \begin{bmatrix} \bar{H}_{k+1} \\ \rho_{k+1}e_{k+1}^T \end{bmatrix}$$

$\|\underline{b} - V_{k+1}H_{k+1}\underline{y}\|_2$

Let $V_{k+1}^c = [V_{k+1} \ \ Q]$
s.t. it's square, orthog.

$$\|\underbrace{(V_{k+1}^c)^T}\,[\underline{b} - V_{k+1}H_{k+1}\underline{y}]\|_2 =$$

$$\|\,\|\underline{b}\|\underline{e}_1 - \underbrace{(V_{k+1}^c)^T V_{k+1}}H_{k+1}\underline{y}\|_2$$

$$= \begin{bmatrix} V_{k+1}^T \\ Q^T \end{bmatrix}V_{k+1} = \begin{bmatrix} I \\ 0 \end{bmatrix}$$

$$\Big\|\,\|\underline{b}\|e_1 - \begin{bmatrix} H_{k+1} \\ 0 \end{bmatrix}y_k\Big\|_2$$

$$= \|\,\|\underline{b}\|e_1 - \underset{\substack{\uparrow \\ k+1 \text{ by } k \text{ least sq.}}}{H_{k+1}}y_k\|_2$$

↓ If A is sym

$$\|\,\|\underline{b}\|e_1 - \underset{\substack{\uparrow \\ \text{tridiag.}}}{T_{k+1}}y_k\|_2$$

Idea 1:
MINRES
picks y_k s.t.

$\|\,\|\underline{b}\|e_1 - T_{k+1}y_k\|$

is minimized.

Idea 2:
CG

Let $T_{k+1} = \begin{bmatrix} T_k \\ \rho e_k^T \end{bmatrix}$ ← k by k then

set $y_k = T_k^{-1}\underline{e}_1 \cdot \|\underline{b}\|$