Lecture 20

* Brief class annoucements, Piazza, HW8

Today: More advanced problems.

How to handle solving multiple linear systems (at the same time, one after another)

How to update the solution of a linear system when the system changes (in an easy-to-describe fashion).

Matrix functions and how they arise from looking at ODEs.

How these apply to counting paths in graphs (Katz centrality) vs. PageRank centrality

How fisher linear discriminant analysis gives rise to a generalized eigenvalue problem.

$n \times n$ $n \times n$

$AX = 0$

$Ax_1 = b_1, Ax_2 = b_2$

$b_2 = \text{function}(x_1)$

$Ax_2 = b \Rightarrow (A + uv^T)y = c$

$\frac{dx}{dt} = Ax(t) \Leftrightarrow x(t) = e^{tA}x(0)$

$A_{22} = \lambda B_{22}$

Multiple RHS

· Given $b_1, \ldots, b_k$ what is $\vec{x}_i$?

· Goal: Solve $Ax_1 = b_1$, $Ax_2 = b_2$

What can the cave do? Go ahead plan:

$AX = I$

$X = A^{-1}$

Multiple RHS

$\begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$

$A_1 x_1 + A_2 x_2 = b_1$

$x_1 = A_1^{-1} b_1 + A_1^{-1} A_2 x_2$

$= \begin{bmatrix} b_1 & A_2 \end{bmatrix} \begin{bmatrix} 1 \\ x_2 \end{bmatrix}$

Multiple RHS

A better plan

What to do?

· Could "\" in Julia/Matlab

What is the runtime/work?

$O(n^3)$ to compute LU, Cholesky, QR

$O(n^2)$ work to solve!

$O(n^3 + n^2 k)$ overall.

If $k > n$ then it dominates the RHS.

Say RHS

$Ax_1 = b_1$

$Ax_2 = b_2 = f(x_1)$

$\vdots$

Example: The inverse power method.

Recall Power method

$x_{k+1} = A x_k$ $\quad \beta_k = \frac{x_k^T A x_k}{x_k^T x_k}$

$x_k \to$ eigenvector of A of largest magnitude λ. λ dominates.

The inverse power method

$y_{k+1} = A^{-1} y_k$ $\quad \beta_k = \frac{1}{y_k^T A y_k}$

If $A \succ 0$, $A = A^T$, $0 < \lambda_1 \le \ldots \le \lambda_n$

$y_k \to$ eigenvector of λ_1

if the unique $(\lambda_1 < \lambda_2)$

A good way to find the closest to λ actually compute A^{-1}.

· Sometimes invert $(A - \lambda I)$ y_k.

Another Case

$\frac{dx}{dt} = Ax(t)$

Given $x(0) = f$, A the goal is to compute $x(T)$, $T \ge 0$.

$\frac{dx}{dt} = \frac{x(t+h) - x(t)}{h}$

$= Ax(t+h)$

$\frac{x(t+h) - x(t)}{h} = Ax(t+h)$

$x(t+h) = (I - hA)^{-1} x(t)$

If we can of $x(0) = \vec{e}_i$ we get row sum at h, T.

In both cases, using a Matrix is a good approach!

· $A \to$ Chol, LU, QR, ... $O(n^3)$ time $(O(n))$

· Solve for each row RHS $\Rightarrow$ $O(n^2)$ time.

In Julia:

$F = \text{lu}(A)$

$F \backslash y$ to solve $Ax = y$

Rank 1 updates

Given A, and a fast way to solve. $A\vec{x} = \vec{b}$

Now, I want you to solve $(A + uv^T)y = b$

If uv^T $\Theta(n)$, then we can $Ay \to Ay + \theta$ (just y).

(Sherman-Morrison-Woodbury formula)

Idea

$Ay + u(v^T y) = c$

$Ay + u\sigma = c$

If σ were known, then

$Ay + u\sigma = c$

$Ay = c - u\sigma$

Suppose σ is fixed

If z solve

$Az = c$

$Az = u$

Then $y = z + \sigma z$

(But we don't know σ ?)

Recall $\sigma = v^T y$

Quiz: What is σ in terms of c, z?

Solve σ Exactly, then use it implicitly

$v^T y = \sigma$

$v^T(c - u\sigma z) = \sigma$

$v^T c + \sigma v^T z = \sigma$

$v^T c = \sigma(1 - v^T z) = \sigma$

$\sigma = \frac{v^T c}{1 - v^T z}$

(says σ in σ)?

With current steps now!

$(A + uv^T)y = c$

$Ay = c - \sigma u$

$Az = c$

$Az = u$ — 2 solves $\Rightarrow A$

$y = z - \sigma z$

$\sigma = v^T y = v^T(c - \sigma z)$

$= v^T z - \sigma v^T z$

$\sigma(1 + v^T z) = v^T z$

$\sigma = \frac{v^T z}{1 + v^T z}$

For rank k

$(A + UV^T)y = c$

U: $n \times k$, V: $n \times k$

$Ay + U \underbrace{V^T y}_{= z} = c$

Using the same idea!

$Ay = c + Uz$

general rank k solve

If A is tridiagonal.

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Matrix functions

Defn: A is symmetric

$$f(A) = V f(\Lambda) V^T$$

where $V \Lambda V^T$ is the eigenpair decomp.

$$f(\Lambda) = \begin{bmatrix} f(\lambda_1) & & \\ & \ddots & \\ & & f(\lambda_n) \end{bmatrix}$$

Why look at this?

$p(A)$ = polynomial of A.

$p(x) = 5x^2 + 6x - 7$

$p(A) = 5A^2 + 6A - 7I$.

or

$$f(A) = V \begin{bmatrix} 5\lambda_1^2 + 6\lambda_1 - 7 & & \\ & \ddots & \\ & & 5\lambda_n^2 + 6\lambda_n - 7 \end{bmatrix} V^T$$

These are the same! Why?

$$A^k = V \Lambda^k V^T$$

These arise when looking at ODEs!

$$\frac{dx}{dt} = A\,x(t)$$

Hopefully you remember

$$\frac{dx}{dt} = ax \quad \text{(scalar case)}$$

Have soln

$$x(t) = e^{at} x(0)$$

$$x'(t) = a e^{at} x(0) = a\,x(t).$$

Also:

$$\frac{dx}{dt} = A\,x(t)$$

$$x(0) = \underline{s}$$

Claim:

$$x(t) = \underbrace{e^{tA}}_{f(A) \text{ where } f(x) = e^{tx}} \underline{s}$$

Pf: Assume $A = A^T$.

$$A = V \Lambda V^T$$

Let

$$y(t) = V^T x(t) \iff x(t) = V y(t)$$

$$\frac{dy}{dt} = V^T \frac{dx}{dt} = V^T A\,x(t) = V^T A V y(t)$$

$$V^T A V = \Lambda$$

$$\frac{dy}{dt} = \Lambda y(t) = \begin{pmatrix} \frac{dy_1}{dt} = \lambda_1 y_1(t) \\ \frac{dy_2}{dt} = \lambda_2 y_2(t) \\ \vdots \end{pmatrix} \Rightarrow y_i(t) = e^{\lambda_i t} y_i(0)$$

$$y(t) = e^{\Lambda t} y(0)$$

$$y(0) = V^T \underline{x}(0)$$

$$\underline{x}(t) = \underbrace{V e^{\Lambda t} V^T}_{= e^{tA}} x(0)$$

Ex: $f(x) = \sqrt{x}$.

$f(A) = \underline{Y}$ ← matrix! when

$$A = Y Y.$$

Ex: $f(x) = \mathrm{sgn}(x) = \begin{cases} +1 & x > 0 \\ -1 & x < 0 \\ 0 & x = 0 \end{cases}$

$$f(x) = 1/x \Rightarrow f(A) = A^{-1}$$

Generalize λ prob

$$A\underline{x} = \lambda B \underline{x}$$

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