Lecture 5

- Office Hours

TA: Wed 11-12pm
Haas G72

Me: Thursday or Friday ✓

(≈ 12:30-1:30)

From SC: 8 mins

$$yy^T x = \alpha y$$ ✓

x When does $A^i \to 0$? ✓

How to check if A is invertible? ✓
det? rank? Eigenvalue?

Phantom Entries in Cholesky for CSC? ✓

x Neumann method uses spectral decomp. ✓
(No!)

Time of operatn

det(A) $O(n!)$ ✓

A^{-1} $O(n^3)$ ✓

- Complexity of x Neumann alg? ✓
- How much do we care about constants? ✓
- Why CSC vs. CSR in Julia? ✓

Neumann Series

$$\boxed{\sum_{i=0}^{\infty} A^i = (I-A)^{-1}} \quad \text{if } \rho(A) < 1$$

geometric series

$$\sum_{i=1}^{\infty} a^i = \frac{1}{1-a} \quad \text{if } |a| < 1$$

$\rho(A)$ = spectral radius.

= largest magnitude λ-val.

= max $|\lambda|$

when λ is an λ-val of A.

$A^i \Rightarrow 0$ iff $\rho(A) < 1$

Why?

if A is symmetric, then $A = V \Lambda V^T$ is the eigenvalue decomp.

$$A^k = (V \Lambda V^T)^k =$$
$$= (V \Lambda \overbrace{V^T)(V}^{I} \Lambda V^T)(V \Lambda V^T)^{k-2}$$

$$A^k = V \Lambda^k V^T.$$

So if $\rho(A) < 1$, then $\Lambda^k \to 0$.

(why?).

If $A \neq A^T$, then let $A = X J X^{-1}$ be the Jordan form.

$$J = \begin{array}{cc} & n \\ n & \begin{bmatrix} \begin{matrix} \lambda_1 & 1 & & 0 \\ & \ddots & \ddots & \\ & & & 1 \\ & & & \lambda_1 \end{matrix} & & 0 \\ & \boxed{\begin{matrix} \lambda_2 & 1 \\ & \lambda_2 \end{matrix}} & 0 \\ 0 & & \ddots \\ & 0 & \end{bmatrix} \end{array}$$

$$A^k = X J^k X^{-1}$$

for solving $\underline{A}\underline{x} = \underline{b}$ w/ Neumann

Alg: let $B = I - A$

$x \leftarrow 0$

for $k = 0$ to ...

$x \leftarrow x + B^k \underline{b}$ $\iff$ $x = Bx + \underline{b}$ (why?)

$$\sum_{k=0}^{\infty} (I-A)^k \underline{b} \to \underline{x}$$

if $x = b +$

then $Bx = Bb$

$B\underline{x} + \underline{b} = b$

$$x^{(k)} = \sum_{i=0}^{k} (I-A)^i \underline{b}$$

⋮

$$Bx^{(k)} = \sum_{\cancel{k}\, i=0}^{k} (I-A)^{i+1} \underline{b}$$

$$Bx^{(k)} + \underline{b} = \sum_{i=0}^{k+1} (I-A)^i \underline{b}$$

$B\underline{b} + B^2\underline{b} + \dots B^k\underline{b}$

$+ B^2\underline{b} + \dots B^{k+1}\underline{b}$

$+ B\underline{b} + B^2\underline{b} \dots + B^{k+1}\underline{b}$

Complexity:

- How big should k be to get a good approx?

- This will depend on $\kappa(A)$ - condition # of A.

$\kappa_2(A) = \|A\|_2 \cdot \|A^{-1}\|_2$.

Complexity:

$O(\text{nnz of } A)$

$O(\text{matvec}(A))$

per iteration

per class

$\det(A) = O(n!)$

No!

$\det(A) = O(n^3)$

$\lambda\text{vals}(A) = O(n^3)$

Matrix-matrix product $= O(n^{\omega})$

$\omega =$ latest paper

$= 2.37\ldots$

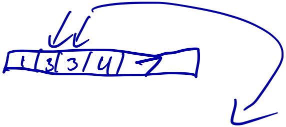
$\omega = \log_2(7)$

from Strassen's Alg.

⊕ practical!

CSC phantom Entry

Colptr



no Entry in column 2.

$$y \overset{=\text{scalar}}{y^T x} = \frac{1}{\alpha} y$$

Assume y is known. then

$$x = \beta y$$

Pick x s.t. $y^T x =$

if $y \geq 0$ (elem-wise), then

$x = \frac{\alpha}{e^T y} e$ will work.

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$$yy^T\left(\frac{\alpha}{e^T y}\,e\right) = \frac{\alpha}{\cancel{e^T y}}\, y\, \cancel{y^T e} = \alpha y$$

Anoth Soln:

$$x = \beta y$$

$$yy^T(\beta y) = (\beta y^T y)\, y = \alpha y$$

$$\Longleftrightarrow$$

$$\alpha = \beta y^T y \Rightarrow$$

$$\beta = \frac{\alpha}{y^T y} \Rightarrow x = \frac{\alpha}{y^T y}\, y$$

$\underline{\alpha}$