RANGE, DIMENSION, AND RANK

Let \( A = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix} \). The range of a set vectors is the vector-space:

\[
\text{range}(A) = \text{span}(a_1, a_2, \ldots, a_n) = \{y \in \mathbb{R}^m \text{ where } y = Ax \text{ for any } x\}
\]

Then

\[
\text{rank}(A) = \dim(\text{range}(A))
\]

where

**dim** (short for dimension) is a property of a vector space which is the minimal number of linearly independent vectors with the same range.

An \( m \times n \) matrix is **full rank** if

\[
\text{rank}(A) = \min(m, n)
\]

AN IMPORTANT THEOREM

THEOREM 1 (Theorem 2.2 in Trefethen and Bau) *If \( m \geq n \), a matrix is full rank if and only if it maps no 2 distinct vectors to the same vector.*

\[
\text{full rank} \iff (Ay_1 = Ay_2 \implies y_1 = y_2)
\]
PROOF

Proof in Picture

Formal proof
THE INVERSE FUNCTION

Let \( b \in \text{range}(A) \). Then the inverse is the function that finds \( x \) such that \( Ax = b \).

*Linear systems of equations*

\[ Ax = b \]

*Our first fundamental problem.*

THE INVERSE MATRIX

If \( A \) is \( n \times n \) and full-rank, then

\[ \mathbb{R}^n \rightarrow \mathbb{R}^n \]
INVERSE PROPERTIES

\[ AA^{-1} = I \text{ and } A^{-1}A = I \]

\[ (AB)^{-1} = B^{-1}A^{-1} \text{ (if both exist)} \]
**EQUIVALENT CONDITIONS FOR LINEAR SYSTEMS**

Let $A$ be $n \times n$

1. $A$ has an inverse
2. $A$ is rank $n$
3. $\text{range}(A) = \mathbb{R}^n$
4. $\text{null}(A) = \{0\}$
5. 0 is not an eigenvector of $A$
6. 0 is not a singular vector of $A$
7. $\det(A) \neq 0$

**SOLVING LINEAR SYSTEMS**

Given the square system

$$Ax = b$$

then

$$x = A^{-1}b$$

but

this is not a good way to compute $x$ unless $A$ is very special.