Let us begin by introducing basic notation for matrices and vectors.

**Matrices**

We’ll use \( \mathbb{R} \) to denote the set of real-numbers and \( \mathbb{C} \) to denote the set of complex numbers.

We write the space of all \( m \times n \) real-valued matrices as \( \mathbb{R}^{m \times n} \). Each

\[
A \in \mathbb{R}^{m \times n} \text{ is } \begin{bmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & \ddots & \vdots \\ A_{m,1} & \cdots & A_{m,n} \end{bmatrix} \text{ where } A_{i,j} \in \mathbb{R}.
\]

Sometimes, I’ll write:

\[
\begin{bmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots \\ a_{m,1} & \cdots & a_{m,n} \end{bmatrix}
\]

instead. With only a few exceptions, matrices are written as *bold, capital* letters. Sometimes, we’ll use a capital greek letter. Matrix elements are written as subscripted, *unbold* letters.

When clear from context, \( A_{i,j} \) is written \( A_{ij} \) instead, e.g. \( A_{11} \) instead of \( A_{1,1} \).

**In class** I’ll usually write matrices with just upper-case letters. If you are unsure if something is a matrix or an element, raise your hand and ask, or *quietly* ask a neighbor.

Another notation for \( A \in \mathbb{R}^{m \times n} \) is

\[ A : n \times n. \]

Sometimes this is nicer to write on the board.

**Vectors**

We write the set of length-\( n \) real-valued vectors as \( \mathbb{R}^n \). Each

\[
x \in \mathbb{R}^n \text{ is } \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \text{ where } x_i \in \mathbb{R}.
\]

Vectors are denoted by *lowercase, bold* letters. As with matrices, elements are subscripted, *unbold* letters. Sometimes, we’ll write vector elements as \( x_i \) or \([x]_i\) or \( x(i)\).
Usually, this choice is motivated by a particular application. Throughout the class, vectors are column vectors.

In class I’ll usually write vectors with just lower-case letters and will try to follow the convention of underlining them.

Scalars

Lower-case greek letters are scalars.

Quick test

Identify the following:

\( f, z_1, x_1, \alpha, \beta, C, C_1, \Sigma, B_{i,j}, b_{i,j} \)

Operations

Transpose Let \( A : m \times n \), then

\[ B : n \times m = A^T \text{ has } B_{i,j} = A_{j,i}. \]

Example \( A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \\ 3 & -1 \end{bmatrix} \) \( A^T = \begin{bmatrix} 2 & 1 \\ 3 & 4 \\ -1 & -1 \end{bmatrix} \)

Hermitian (Also called conjugate transpose) Let \( A \in \mathbb{C}^{m \times n} \), then

\[ B \in \mathbb{C}^{n \times m} = A^* = A^H \text{ has } B_{i,j} = \overline{A}_{j,i}. \]

Example \( A = \begin{bmatrix} 2 & 3 \\ i & 3 \\ 3 & -i \end{bmatrix} \) \( A^* = \begin{bmatrix} 2 & 3 \\ -i & i \\ 3 & -i \end{bmatrix} \)

Addition Let \( A : m \times n \) and \( B : m \times n \), then

\[ C : m \times n = A + B \implies C_{i,j} = A_{i,j} + B_{i,j}. \]

Example \( A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \\ 3 & -1 \end{bmatrix} , B = \begin{bmatrix} 1 & -1 \\ 2 & 3 \\ -1 & 0 \end{bmatrix} \) \( A + B = \begin{bmatrix} 3 & -2 \\ 5 & 7 \\ 2 & -1 \end{bmatrix} \).

Scalar Multiplication Let \( A : m \times n \) and \( \alpha \in \mathbb{R} \), then

\[ C : m \times n = \alpha A + B \implies C_{i,j} = \alpha A_{i,j}. \]

Example \( A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} , 5A = \begin{bmatrix} 10 & 15 \\ 5 & 20 \end{bmatrix} \)

Matrix Multiplication Let \( A : m \times n \) and \( B : n \times k \), then

\[ C : m \times k = AB \implies C_{i,j} = \sum_{r=1}^{n} A_{i,r} B_{r,j}. \]

Matrix-vector Multiplication Let \( A : m \times n \) and \( x \in \mathbb{R}^n \), then

\[ c \in \mathbb{R}^m = Ax \implies c_i = \sum_{j=1}^{n} A_{i,j} x_j. \]
This operation is just a special case of matrix multiplication that follows from treating \( \mathbf{x} \) and \( \mathbf{c} \) as \( n \times 1 \) and \( m \times 1 \) matrices, respectively.

**Vector addition, Scalar vector multiplication** These are just special cases of matrix addition and scalar matrix multiplication where vectors are viewed as \( n \times 1 \) matrices.

**Partitioning**

It is often useful to represent a matrix as a collection of vectors. In this case, we write

\[
\mathbf{A} : m \times n = [\mathbf{a}_1 \ \mathbf{a}_2 \ \cdots \ \mathbf{a}_n]
\]

where each \( \mathbf{a}_j \in \mathbb{R}^m \). *This form corresponds to a partition into columns.*

Alternatively, we may wish to partition a matrix into rows.

\[
\mathbf{A} : m \times n = \begin{bmatrix}
\mathbf{r}_1^T \\
\mathbf{r}_2^T \\
\vdots \\
\mathbf{r}_m^T
\end{bmatrix}
\]

Here, each \( \mathbf{r}_i \in \mathbb{R}^n \).

Using the column partitioning:

\[
\mathbf{A}\mathbf{x} = [\mathbf{a}_1 \ \mathbf{a}_2 \ \cdots \ \mathbf{a}_n] \begin{bmatrix}
\mathbf{x}_1 \\
\mathbf{x}_2 \\
\vdots \\
\mathbf{x}_n
\end{bmatrix} = \sum_j \mathbf{x}_j \mathbf{a}_j.
\]

And with the row partitioning:

\[
\mathbf{A}\mathbf{x} = \begin{bmatrix}
\mathbf{r}_1^T \\
\mathbf{r}_2^T \\
\vdots \\
\mathbf{r}_m^T
\end{bmatrix} \mathbf{x} = \begin{bmatrix}
\mathbf{r}_1^T \mathbf{x} \\
\mathbf{r}_2^T \mathbf{x} \\
\vdots \\
\mathbf{r}_m^T \mathbf{x}
\end{bmatrix}.
\]

Another useful partitioned representation of a matrix is into blocks:

\[
\mathbf{A} = \begin{bmatrix}
\mathbf{A}_{1,1} & \mathbf{A}_{1,2} \\
\mathbf{A}_{2,1} & \mathbf{A}_{2,2}
\end{bmatrix}
\]

or

\[
\mathbf{A} = \begin{bmatrix}
\mathbf{A}_{1,1} & \mathbf{A}_{1,2} & \mathbf{A}_{1,3} \\
\mathbf{A}_{2,1} & \mathbf{A}_{2,2} & \mathbf{A}_{2,3} \\
\mathbf{A}_{3,1} & \mathbf{A}_{3,2} & \mathbf{A}_{3,3}
\end{bmatrix}.
\]

Here, the sizes “just have to work out” in the vernacular. Formally, all \( \mathbf{A}_{i,j} \) must have the same number of rows and all \( \mathbf{A}_{-j} \) must have the same number of columns.