Let us begin by introducing basic notation for matrices and vectors.

Matrices

We’ll use $\mathbb{R}$ to denote the set of real-numbers and $\mathbb{C}$ to denote the set of complex numbers.

We write the space of all $m \times n$ real-valued matrices as $\mathbb{R}^{m \times n}$. Each

$$A \in \mathbb{R}^{m \times n} \text{ is } \begin{bmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & \ddots & \vdots \\ A_{m,1} & \cdots & A_{m,n} \end{bmatrix} \text{ where } A_{i,j} \in \mathbb{R}.$$  

Sometimes, I’ll write:

$$\begin{bmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots \\ a_{m,1} & \cdots & a_{m,n} \end{bmatrix}$$

instead. With only a few exceptions, matrices are written as bold, capital letters. Sometimes, we’ll use a capital greek letter. Matrix elements are written as sub-scripted, unbold letters.

When clear from context, $A_{i,j}$ is written $A_{ij}$ instead, e.g. $A_{11}$ instead of $A_{1,1}$.

In class I’ll usually write matrices with just upper-case letters. If you are unsure if something is a matrix or an element, raise your hand and ask, or quietly ask a neighbor.

Another notation for $A \in \mathbb{R}^{m \times n}$ is

$$A : n \times n.$$  

Sometimes this is nicer to write on the board.

Vectors

We write the set of length-$n$ real-valued vectors as $\mathbb{R}^n$. Each

$$x \in \mathbb{R}^n \text{ is } \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \text{ where } x_i \in \mathbb{R}.$$  

Vectors are denoted by lowercase, bold letters. As with matrices, elements are sub-scripted, unbold letters. Sometimes, we’ll write vector elements as $x_i$ or $[x]_i$ or $x(i)$.

Usually, this choice is motivated by a particular application. Throughout the class, vectors are column vectors.

In class I’ll usually write vectors with just lower-case letters and will try to follow the convention of underlining them.
 Scalars

Lower-case greek letters are scalars.

Quick test

Identify the following:

\(f, z_1, x_1, \alpha, \beta, C, C_1, \Sigma, B_{i,j}, b_{i,j}\)

Operations

Transpose

Let \(A : m \times n\), then

\[ B : n \times m = A^T \text{ has } B_{i,j} = A_{j,i}. \]

Example

\[
A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \\ 3 & 7 \\ -1 & 0 \\
\end{bmatrix}, \quad A^T = \begin{bmatrix} 2 & 1 & 3 & -1 \\ 3 & 4 & 7 & 0 \\
\end{bmatrix}
\]

Hermitian (Also called conjugate transpose) Let \(A \in \mathbb{C}^{m \times n}\), then

\[ B \in \mathbb{C}^{n \times m} = A^* = A^H \text{ has } B_{i,j} = \overline{A_{j,i}}. \]

Example

\[
A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \\ -1 & 3 \
\end{bmatrix}, \quad A^* = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 4 & i \\
\end{bmatrix}
\]

Addition

Let \(A : m \times n\) and \(B : m \times n\), then

\[ C : m \times n = A + B \implies C_{i,j} = A_{i,j} + B_{i,j}. \]

Example

\[
A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \\ -1 & 3 \\
\end{bmatrix}, \quad B = \begin{bmatrix} 1 & -1 \\ 2 & 3 \\ -1 & 1 \\
\end{bmatrix}, \quad A + B = \begin{bmatrix} 3 & 2 \\ 3 & 7 \\ 2 & 4 \\
\end{bmatrix}
\]

Scalar Multiplication

Let \(A : m \times n\) and \(\alpha \in \mathbb{R}\), then

\[ C : m \times n = \alpha A + B \implies C_{i,j} = \alpha A_{i,j}. \]

Example

\[
A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \\
\end{bmatrix}, \quad 5A = \begin{bmatrix} 10 & 15 \\ 5 & 20 \\
\end{bmatrix}
\]

Matrix Multiplication

Let \(A : m \times n\) and \(B : n \times k\), then

\[ C : m \times k = AB \implies C_{i,j} = \sum_{r=1}^{n} A_{i,r}B_{r,j}. \]

Matrix-vector Multiplication

Let \(A : m \times n\) and \(x \in \mathbb{R}^n\), then

\[ \mathbf{c} \in \mathbb{R}^m = Ax \implies c_i = \sum_{j=1}^{n} A_{i,j}x_j. \]

This operation is just a special case of matrix multiplication that follows from treating \(x\) and \(c\) as \(n \times 1\) and \(m \times 1\) matrices, respectively.

Vector addition, Scalar vector multiplication

These are just special cases of matrix addition and scalar matrix multiplication where vectors are viewed as \(n \times 1\) matrices.
Partitioning

It is often useful to represent a matrix as a collection of vectors. In this case, we write

$$A : m \times n = [a_1 \ a_2 \ \cdots \ a_n]$$

where each $$a_j \in \mathbb{R}^m$$. This form corresponds to a partition into columns.

Alternatively, we may wish to partition a matrix into rows.

$$A : m \times n = \begin{bmatrix} r_1^T \\ r_2^T \\ \vdots \\ r_m^T \end{bmatrix}$$

Here, each $$r_i \in \mathbb{R}^n$$.

Using the column partitioning:

$$Ax = [a_1 \ a_2 \ \cdots \ a_n] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \sum_j x_j a_j.$$ 

And with the row partitioning:

$$Ax = \begin{bmatrix} r_1^T \\ r_2^T \\ \vdots \\ r_m^T \end{bmatrix} x = \begin{bmatrix} r_1^T x \\ r_2^T x \\ \vdots \\ r_m^T x \end{bmatrix}.$$ 

Another useful partitioned representation of a matrix is into blocks:

$$A = \begin{bmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{bmatrix}$$

or

$$A = \begin{bmatrix} A_{1,1} & A_{1,2} & A_{1,3} \\ A_{2,1} & A_{2,2} & A_{2,3} \\ A_{3,1} & A_{3,2} & A_{3,3} \end{bmatrix}.$$ 

Here, the sizes “just have to work out” in the vernacular. Formally, all $$A_{i,j}$$ must have the same number of rows and all $$A_{i,j}$$ must have the same number of columns.