We now illuminate some of the relationships between matrix computations and linear algebra.

Why is this stuff important? The important bit is the concept of the rank of a matrix. This gives the dimension of the vector-space associated with the matrix. So it’s worth reviewing up to the point of rank.

Sets of vectors

**Linearly independent** A set of vectors \( \{x_1, \ldots, x_k\} \) in \( \mathbb{R}^n \) is called linearly independent if

\[
\sum_{i=1}^{k} \alpha_i x_i = 0
\]

implies \( \alpha_i = 0 \) all \( i \).

*Examples* The vectors \( x_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \) and \( x_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \) are linearly independent. This can be verified by showing that the system of equations:

\[
\alpha_1 + 2\alpha_2 = 0 \quad \text{and} \quad 2\alpha_1 + 3\alpha_2 = 0
\]

only has the solution \( \alpha_1 = \alpha_2 = 0 \). However, the vectors \( x_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \) and \( x_2 = \begin{bmatrix} 2 \\ 4 \end{bmatrix} \)

are not linearly independent because \( 2x_1 - x_2 = 0 \).

*As a matrix* The property of being linearly independent is easy to state as a matrix. Suppose that \( X \) is an \( n \times k \) matrix where \( x_i \) is the \( i \)th column:

\[
X = \begin{bmatrix} x_1 & \cdots & x_k \end{bmatrix}.
\]

Then the set of vectors is linearly independent if \( Xa = 0 \) implies that \( a = 0 \).

*Span* (not spam) The span of a set of vectors is the set of all linear combinations.

\[
\text{span}(x_1, \ldots, x_k) = \left\{ \sum_{i=1}^{k} \alpha_i x_i, \alpha_i \in \mathbb{R} \right\}.
\]

Subspaces

Defining a vector spaces is best left to Wikipedia:

- **Vector space**

Suffice it to say that that the set \( \mathbb{R}^n \) is a vector-space with the field of real-numbers as scalars.

A subset \( V \subset \mathbb{R}^n \) is called a *subspace* if it also satisfies the properties of being a vector-space itself.

*Example* Let \( V = \{ \alpha x, \alpha \in RR \} \) for some vector \( x \in \mathbb{R}^n \). Then \( V \) is a subspace of \( \mathbb{R}^n \).

*Spans and subspaces* The example we just saw shows that \( \text{span}(x) \), the span of a single vector, is a subspace. This is true in general: \( \text{span}(x_1, \ldots, x_k) \) is a subspace.
**Linearly independent spans** Let \( x_1, \ldots, x_k \) be linearly independent. Then for \( b \in \text{span}(x_1, \ldots, x_k) \), there exists a unique set of \( \alpha_i \)'s such that \( b = \sum_{i=1}^{k} \alpha_i x_k \). As a matrix, this is saying that the system of equations:

\[
b = Xa
\]

has a unique solution \( a \) where

\[
X = [x_1, \ldots, x_k].
\]

**Subspaces to bases and dimensions** For any subspace \( V \subseteq \mathbb{R}^n \), we can find always find a set \( S \) of linearly independent vectors \( S = \{x_1, \ldots, x_k\} \) such that \( V = \text{span}(x_1, \ldots, x_k) \). We call any such set a *basis* for the subspace \( V \).

**IMPORTANT** Any basis for a subspace always has the same number of vectors. Thus, the number of vectors in a subspace is a unique property of a vector space and is the dimension of the vector-space.

This ends our discussion of subspaces. Now we’ll see how we can use subspaces to discuss matrices.

**Matrices to subspaces**

Given a matrix \( A \in \mathbb{R}^{m \times n} = [a_1, \ldots, a_n] \).

**Range** The range of a matrix is the subspace:

\[
\text{range}(A) = \{y \in \mathbb{R}^m : y = Ax \text{ for all } x \in \mathbb{R}^n\}.
\]

Note that

\[
\text{range}(A) = \text{span}\{a_1, \ldots, a_n\}.
\]

So the range is just one particular span of a set of vectors.

**Rank**

Perhaps the most important thing in these notes is the concept of rank. At this point, rank is simple.

\[
\text{rank}(A) = \dim(\text{range}(A))
\]

That is, the rank of \( A \) is the dimension of the subspace given by the range of \( A \). This property is fundamentally important.

For instance, if \( A \in \mathbb{R}^{m \times n} \) with \( m \geq n \) and \( \text{rank}(A) = n \), then we know that

\[
A = [a_1, \ldots, a_n]
\]

has a set of linearly independent column vectors!

**Example** Here’s where we can use some of our matrix algebra to prove a statement. Let \( P \) be an \( n \times n \) permutation matrix. Show that \( \text{rank}(AP) = \text{rank}(A) \).

**Proof Sketch** A permutation matrix just reorders the columns of the matrix. This won’t change anything in the range of \( A \). So the set of vectors in the range of \( A \) won’t change. Thus, the dimension of that vector space won’t change.

**Key question** How do we compute rank?

**Answer** Use a matrix decomposition!
Useful matrix decompositions

Let \( A \in \mathbb{R}^{m \times n} \) be a matrix. The following are matrix decompositions exist for any matrix:

1. \( A = QR \) where \( Q \) is \( m \times m \) and orthogonal, and \( R \) is \( m \times n \) and upper-triangular.

2. \( A = U \Sigma V^T \) where \( U \) is \( m \times m \) and orthogonal, \( V \) is \( n \times n \) and orthogonal, and \( \Sigma \) is \( m \times n \) and diagonal, with diagonal entries \( \sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n \geq 0 \).
   (That is, sorted in decreasing order and non-negative.)

3. \( A = PLUQ \) where \( P \) and \( Q \) are permutation matrices and \( L \) and \( U \) are lower and upper triangular.

These decompositions expose the rank of a matrix in various ways. For instance, the number of entries on the diagonal of \( A \) that are non-zero is equal to the rank of the matrix.