In this lecture, we’ll look into the inverse of a matrix, and find out which matrices are invertible. These notes roughly follow Trefethen and Bau, section 1.

Background

Let \( A \in \mathbb{R}^{m \times n} = [a_1 \cdots a_n] \). Recall that

\[
\begin{align*}
\text{range}(A) &= \text{span}(a_1, \ldots, a_n), \\
\text{rank}(A) &= \dim(\text{range}(A)).
\end{align*}
\]

We call \( A \in \mathbb{R}^{m \times n} \) full rank if \( \text{rank}(A) = \min(m, n) \).

Full rank matrices

Full rank matrices have the important property that they give rise to one-to-one maps between \( \mathbb{R}^n \) and \( \mathbb{R}^m \). Let’s show this.

**Theorem** (From Trefethen and Bau, Theorem 1.2) Let \( m \geq n \), a matrix is full rank if and only if it maps no two distinct vectors to the same vector.

**Proof** If a matrix is full rank, then it has \( n \) linearly independent column vectors. These vectors are a basis for \( \text{range}(A) \). This, in turn, implies that any vector in \( \text{range}(A) \) has a unique representation in this basis. (If not, then \( A c_1 = A c_2 \) and so \( A \) has linearly dependent columns, which it can’t!) Thus, any vector \( Ay \) corresponds to a unique \( y \).

We also have to prove the reverse direction, but this is easier to prove via the contrapositive. If \( A \) is not full rank, then it’s columns are linearly dependent. Hence, there exists a vector \( c \) such that \( A c = 0 \). Let \( y \) be any vector in \( \mathbb{R}^n \), then \( Ay = A(y + c) \) and so we have two distinct vectors that give us the same result.

There’s a great picture I could put here, but it’s too tricky. The point is that we have a one-to-one map between \( \mathbb{R}^n \) and \( \text{range}(A) \), which is a subset of \( \mathbb{R}^m \) when \( m \geq n \). Because this map is one-to-one, it’s invertible! So we can take any vector \( b \in \text{range}(A) \) and find

\[
Ax = b
\]

for some \( x \in \mathbb{R}^n \).

Linear systems

It’s worth repeating this equation because it’s so fundamental to the rest of the class – and the entire field.

We call

\[
Ax = b
\]

a linear system of equations.

Usually these are defined with squares matrices \( A \).
Square, full rank matrices

Let \( A \in \mathbb{R}^{n \times n} \) be a full-rank matrix. What we’ve shown above is that any vector in \( \mathbb{R}^n \) can be written as \( Ax \) for some unique \( x \).

Thus, we can find the following \( n \) vectors:

\[
Ax_i = e_i, \quad i = 1, \ldots, n.
\]

If we write this as a matrix equation, we have:

\[
AX = I.
\]

The matrix \( X \) is called the inverse and usually written \( A^{-1} \).

The matrix inverse

We’ve shown that \( A \in \mathbb{R}^{n \times n} \) and full rank has an inverse \( A^{-1} \) such that

\[
AA^{-1} = I.
\]

Let’s study a few properties of this inverse.

First, does \( A^{-1}A = I \) too? We’ll show this is the case. Let \( AX = I \) and let \( YA = I \). Then

\[
YAX = (YA)X = X,
\]

but also

\[
YAX = Y(mAX) = Y.
\]

Thus, \( X = Y \).

Second, \( (AB)^{-1} = B^{-1}A^{-1} \) assuming that \( A \) and \( B \) are square. The standard way to check that you have the inverse of a particular matrix is just to show that it satisfies \( AX = I \). In this case:

\[
(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = (AA^{-1} = I).
\]

where we’ve canceled all the inverse pairs represented by parentheses.

When is a matrix full rank?

The following set of statements from Trefthen and Bau helps to characterize when a matrix has an inverse. Let \( A \in \mathbb{R}^{n \times n} \), these statements are all equivalent to each other:

1. \( A \) has an inverse \( A^{-1} \)
2. \( A \) has rank \( n \)
3. range(\( A \)) = \( \mathbb{R}^n \).
4. (not fully covered) null(\( A \)) = \{0\} (null is the nullspace).
5. (not fully covered) 0 is not an eigenvalue of \( A \)
6. (not fully covered) 0 is not a singular value of \( A \)
7. (not fully covered) \( \det(A) \neq 0 \)
Solving a linear system

Let $A \in \mathbb{R}^{n \times n}$ be full-rank. Then the linear system:

$$Ax = b$$

has solution

$$x = A^{-1}b.$$  

Be warned, this is not a good way to find $x$ unless $A$ is very special. In this class, we will see many superior procedures to find $x$ that satisfies this linear system. A good way to demonstrate that you have not learned the material is to utilize this idea in your programs.