Homework 9

Please answer the following questions in complete sentences in a typed, clearly prepared manuscript and submit the solution by the due date on Blackboard (Friday, November 20th, 2015, 5pm, with all the usual extensions...)

Problem 0: Homework checklist

- Please identify anyone, whether or not they are in the class, with whom you discussed your homework. This problem is worth 1 point, but on a multiplicative scale.
- Make sure you have included your source-code and prepared your solution according to the most recent Piazza note on homework submissions.

Problem 1: Compare CG as described in class to the standard CG iteration

One of the most remarkable results in iterative methods is how straightforward many algorithms can be derived, and how these straightforward derivations give rise to beautifully simple implementations. In this problem you'll explore this property with the conjugate gradient method.

1. Implement a CG method based on the Lanczos method. That is, implement the Lanczos method to compute the matrix $T_k$ and $V_k$. Then solve $T_k y^{(k)} = e_1$ and compute $x = ||b||_2 V_k y^{(k)}$. (All this notation is taken from the CG handout from class, here https://www.cs.purdue.edu/homes/dgleich/cs515-2015/lectures/conjugate-gradient.pdf). Show the code for your implementation. At each step, you should compute the normalized residuals.

2. Compare the first 25 residuals from your Lanczos-based CG code with the standard implementation of CG from: http://www.cs.purdue.edu/homes/dgleich/cs515-2015/matlab/cg.m for the linear system

   \[
   n = 100; \\
   on = ones(n,1); \\
   A = spdiags([-2*on 4*on -2*on],-1:1,n,n); \\
   b = ones(n,1); 
   \]

3. Using the cg.m function, look at how many iterations are required for CG to converge to a tolerance of $10^{-8}$ for the matrix in the last part. Determine how this scales with $n$.

Problem 2: Orthogonality of Lanczos

Let $\lambda_1 = 0.1$ and $\lambda_n = 100$, $\rho = 0.9$, $n = 30$.
Consider the $n$-by-$n$ matrix with diagonal elements

\[
d_i = \lambda_1 + (i - 1)/(n - 1)(\lambda_n - \lambda_1)\rho^{n-i}.
\]

(This is called the Strakoš matrix.)
1. Implement the Lanczos method starting from a random vector $v_1$ and the vector $v_1 = e/\sqrt{n}$ and then plot the quantity $\log(\|V_k^T V_k - I\| + 10^{-20})$ for $k = 1$ to 30. Describe what you SHOULD find and what you actually find. Do your results depend on the starting vector?

2. Plot $\log(|v_1^T v_k| + 10^{-20})$ for the $k = 1$ to 30. Also plot $\log(|v_{k-2}^T v_k| + 10^{-20})$ for $k = 3$ to 30.

3. What is $\beta_{31}$? What should it be?

4. Plot $\log(\|AV_k - V_{k+1}T_{k+1}\| + 10^{-20})$ for $k = 1$ to 60.