Homework 7

Please answer the following questions in complete sentences in a typed, clearly prepared manuscript and submit the solution by the due date on Blackboard (Friday, November 6th, 2015, 5pm, with all the usual extensions...)

Problem 0: Homework checklist

- Please identify anyone, whether or not they are in the class, with whom you discussed your homework. This problem is worth 1 point, but on a multiplicative scale.
- Make sure you have included your source-code and prepared your solution according to the most recent Piazza note on homework submissions.

Problem 1: Prove or disprove

People seem to like these questions!

1. The eigenvalues of an $n \times n$ real-valued matrix are always real.
2. The solution to $(A^T A + \gamma I)x = b$ is unique for any $\gamma > 0$.
3. Every $n \times n$ matrix can be written as the sum of two non-singular matrices.
4. An $n \times n$ real-valued matrix has an orthogonal set of eigenvectors.

Problem 2: The power method, and beyond!

We’ll show that the humble power-method is really rather more interested than I alluded to in class! It’s a flexible starting point that serves as the basis for almost all of the eigenvalue algorithms.

1. (Warm up) Consider the power method as a Matlab code:

```matlab
maxit = 1000;
for i=1:maxit
    x = A*x;
    x = x/norm(x);
    lam = x'*A*x;
end
```

Let $x^{(i)}$ be the value of $x$ at the start of the $i$th instance of this loop. And let $\lambda^{(i)}$ be the value of $\text{lam}$ at the start of the $i$th iteration. Suppose that $x$ and $\lambda$ are the true eigenvector, eigenvalue pair that we are converging too. If $\|x - x^{(i)}\| = \varepsilon$, show that: $|\lambda - \lambda^i|$ is $O(\varepsilon^2)$ (where the constant may depend on the matrix $A$).

2. (Warm up) Show that if $x$ is an eigenvector of $A$ with eigenvalue $\lambda$, then $x$ is an eigenvector of $A^{-1}$. What is the eigenvalue?

3. Let $A$ have a complete set of eigenvalues $\lambda_i \neq \lambda_j$, for all $i \neq j$ and associated eigenvectors $x_i$. For the sake of simplicity, let’s suppose that all the eigenvalues are real. Suppose we run the following Matlab code:
maxit = 1000;
[L,U,P] = lu(A);
for i=1:maxit
    x = U\(L\((P*x));
    x = x/norm(x);
    lam = x'*A*x;
end

Does this converge? If so, what does it converge to, and how fast does it converge?

4. Suppose you were in a parallel linear algebra class, taught by another professor, and you were answering the following question:

Make the power method converge faster to any eigenvalue, eigenvector pair!
Dense linear algebra is cheap. Work in pairs.

Your friend says that you should look at this algorithm:

maxit = 1000;
for i=1:maxit
    x = (A - lam*eye(size(A,1)))\x);
    x = x/norm(x);
    lam = x'*A*x;
end

This algorithm is called “Rayleigh Quotient Iteration” and the standard result is that \(x^{(k)}\) converges to an eigenvector cubically. That is, if \(|x^{(k)} - x| = \varepsilon\), then \(|x^{(k+1)} - x| = O(\varepsilon^3)\). Look up this result in a textbook, or on the web, and explain the proof in your own words.

5. Implement this method. Do you observe cubic convergence if you try to find an eigenvalue, eigenvector pair for a symmetric matrix?

Problem 3: PageRank and the power method

In class, I derived PageRank as the solution of the linear system:

\[(I - \alpha P)x = (1 - \alpha)v\]

where \(P\) is a column-stochastic matrix, \(0 \leq \alpha < 1\) and \(v\) is a non-negative vector whose elements sum to 1.

1. (Warm up) Use eigenvalue properties of \((I - \alpha P)\) to show that it is a non-singular matrix. (Hint: recall that \(\rho(P) \leq \|P\|\) for any norm. Is there a norm where we know \(\|P\|\) has a known value?)

2. The standard definition of PageRank is as the largest eigenvector of the matrix:

\[M = \alpha P + (1 - \alpha)v^T\]

normalized to have sum one (instead of the standard 2-norm scaling). Consider the power method, without normalization, applied to \(M\) starting from the vector \(v\):

\[x^{(k+1)} = Mx^{(k)} = M^{k+1}v.\]

Show that the iterates from this method are always non-negative and sum to 1.
3. Use this fact to simplify the iteration and show that the power method will converge to the solution of the linear system \((I - \alpha P)x = (1 - \alpha)v\). You should get the same iteration that I discussed in class.


5. Use the power method to compute the PageRank vector \(x\) for \(P\) using \(\alpha = 0.85\), \(v_i = 1/n\) where \(n\) is the size of the matrix. In Matlab

```matlab
n = size(P,1);
v = ones(n,1)./n;
alpha = 0.85;
```

What is \(x(1)\)? Use the following code to sort the vector and output the top 27 entries:

```matlab
[ignore p] = sort(x,'descend');
urls(p(1:27))
```

Report the output from this code.