Lecture 8

- Finite linear function Spaces
- Why polynomials?
- Weierstrass Approx Theorem
- Best Approx in general
  - $\Rightarrow$ Issues
- Orthogonal bases
- Sequences of approx

Finite function Spaces

Let $\pi_i(t)$ be a collection of linearly indep. functions then we often look at

$$\Phi = \{ c_1\pi_1 + c_2\pi_2 + \dots + c_n\pi_n \mid c_i \in \mathbb{R} \}$$

Eg $\pi_i = t^{i-1}$ gives monomials.

→ A set of functions is linearly independent if

$\sum c_i \pi_i = 0 \iff c_i = 0$

- Only one way to represent 0 function w/ all coeffs. = 0.

What π_i to use?

→ polynomials

→ trig functions (for periodic f)

Why polys?

- they are simple!

  ↳ monomials are "Easy" to work with!

Why poly #2

- Weierstrass Approx Theorem:

Let f be continuous in $[-1,1]$.

Let $\varepsilon > 0$ be arbitrary.

Then there exists p (a polynomial) s.t.

$$\|p - f\|_\infty \le \varepsilon$$

→ $\|p - f\|_\infty = \max_{x \in [-1,1]} |p(x) - f(x)|$.

Pf (big picture):

(extremely smooth)

$$f \longleftrightarrow \hat{f}$$

morphing with heat flow.

- Formally setup a heat flow PDE that morphs.

↳ Heat flow gives us Convergent Taylor series c small "step"

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→ Chapter 6 in ATAP by Trefethen!

Back to Best Approx... with Least Squares

Let Φ be a finite function space.

$$\Phi = \{ \sum_{i=1}^{n} c_i \cdot \pi_i \mid c_i \in \mathbb{R})$$

We want to find

$$\phi \in \Phi \text{ s.t. }$$

$\| f - \phi \|_2$ is as small as possible.

→ Equiv. find ϕ s.t.

$E = \|f - \phi\|_2^2$ is as small as possible.

$\phi = \sum c_i \pi_i$

$E = \|f - \sum c_i \pi_i\|_2^2$

$= \int_a^b (f - c_1\pi_1 - c_2\pi_2 - \dots c_n\pi_n)^2 dx$

$$\begin{array}{c|cccc} & f & \pi_1 & - & \pi_n \\ \hline f & & & & \\ \pi_1 & & & & \\ | & & & & \\ \pi_n & & & & \end{array}$$

→ $\int_a^b \Big(f^2 - c_1 f\pi_1 - c_2 f\pi_2 - \dots$

$- c_1 f\pi_1 + c_1^2\pi_1^2 + \dots$

$- c_2 f\pi_2 - c_1 c_2 \pi_1\pi_2 + c_2^2\pi_2^2 +$

Sh

dx = break into terms

$\int_a^b f^2 dx - \int_a^b f\pi_1 dx - \dots$

these are all terms like

$\int_a^b fg\, dx$

ld have
been
positive!

-)

!

-

Inner Products of functions

$\int_a^b f(x)g(x)\,dx$ or

$\int f(t)g(t)\,d\lambda(t)$ or

$\int fg\,d\lambda(t)$ or

$\boxed{(f,g)}$ simplest!

These inner-products have nice properties!

- $(f,g) = (g,f)$ Symmetric
- $(\alpha f, g) = \alpha (f,g)$ Homogeneous
- $(u+v,w) = (u,w) + (v,w)$ Additive
- $(u,u) \geq 0$ and $(u,u) = 0 \Leftrightarrow u = 0$ pos. def.

$\Rightarrow$ property of linearity!

$$E^2 = \|f - \phi\|_2^2 = (f-\phi, f-\phi) = (f,f) - 2(f,\phi) + \boxed{(\phi,\phi)}$$

Assim $\phi = \sum c_i \pi_i$

$$= (f,f) - 2\sum c_i (f,\pi_i) + \sum_{ij} \underbrace{(c_i \pi_i, c_j \pi_j)}_{= c_i c_j (\pi_i, \pi_j)}$$

$E(c)^2$ is a quadratic eq. in $\underline{c}$

$\Longleftrightarrow$

vect

$$E(c)^2 = \gamma + c^T \underline{v} + \underline{c}^T A \underline{c}$$

matrix

$$v_i = -\underline{2}(f, \pi_i)$$

$$A_{ij} = (\pi_i, \pi_j)$$

Just take

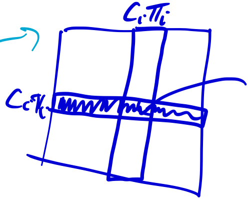
$$\Rightarrow \frac{\partial E^2}{\partial c_i} = 0$$

to find a minimizer!

$$\frac{\partial E^2}{\partial c_i} = \frac{\partial}{\partial c_i}\left[(f,f) - 2\sum c_i (f, \pi_i) + \sum_{ij} \underbrace{(c_i \pi_i, c_j \pi_j)}_{= c_i c_j (\pi_i, \pi_j)} \right]$$

$$= 0$$

$$-2 (f, \pi_i$$



$$\Rightarrow 2 \frac{\partial}{\partial c_i} \sum_{j \neq i} c_i c_j c$$

$$+$$

$$\frac{\partial}{\partial c_i} c_i^2 (\pi_i, \pi$$

$)$

$\pi_i, \pi_j)$

$i\ )$

$$2 \sum_{j \neq i} c_j (\pi_i, \pi_j) + 2 c_i (\pi_i, \pi_i)$$

$$\frac{\partial}{\partial c_i}(E^2) = -2(f, \pi_i) + 2 \sum_j c_j (\pi_i, \pi_j) = 0$$

$$\boxed{A\underline{c} = \underline{b}}$$

where

$A_{ij} = (\pi_i, \pi_j)$

$b_i = (f, \pi_i)$

Called the normal eq. for best least sq. approx. of f!

Issue of Normal Eq.

$$A \underline{c} = \underline{b}$$

$$A_{ij} = (\pi_i, \pi_j)$$

$$b_i = (\pi_i, f)$$

1. A's often

$\Rightarrow$ any

Cause hug

2. If we Φ_n

n

we need to

Φ_m

ill-conditioned

"mistake" in $\underline{b}$

e Errors in $\underline{c}$.

vs Φ_m

$\geq m$

resolve Even if

c Φ_1.

These issues go away if the function

π_i are orthogonal!

If $(\pi_i, \pi_j) = 0$

then π_i, π_j are orthogonal!

So we

$(\pi_i,$

if

Ex. $f = 1$

$\int_{-1}^{1} fg \, dt$

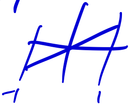
here

$\pi_j) = 0$

$i \neq j$.

$g(t) = t$

$= \int_{-1}^{1} t = 0$.



$h(t) = \frac{1}{2}(3t^2 - 1)$

$\int_{-1}^{1} fh \, dt = 0$

$\int_{-1}^{1} fg \, dt = 0$.

These are the first 3 Legendre polynomials.

If

and

Then

A_c

C

$$\Phi = \{c_1 \pi_1 + \dots + c_n \pi_n \mid c_i \in \mathbb{R}\}$$

$$(\pi_i, \pi_j) = 0. \quad (i \neq j)$$

$$A_{ij} = 0 \quad i \neq j$$

$\Rightarrow \underline{A}$ is diag.

$$c_i = \frac{(\pi_i, f)}{(\pi_i, \pi_i)}$$

If we add:

$\pi_{n+1}, \dots \pi_{n+2}, \dots$

then all c_i before stay the same!

It's often better
to use:

$$c_i = \frac{1}{(\pi_i, \pi_i)} \left(f - \sum_{k=1}^{i-1} c_k \pi_k, \pi_i\right)$$

Why?

$$|\pi_i|, |f| = O(|f|)$$

$c_i \to 0$ ("Convergence"

$\pi_i \to 0$

$\simeq 0$

$\approx 0.$

$$c_i := \frac{(\pi_i, f)}{(\pi_0, \pi_i)}$$

$= 0$

large terms but they cancel!

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