Lecture 7

- Linear Function Spaces
- Best Approximation Problem
- Function Norms!
- Weighted Function Norms
- Examples in Julia

A linear function space

$$\Phi$$

is an finite set of function from $\mathbb{R} \to \mathbb{R}$

S.t. • $f_1, f_2 \in \Phi \Rightarrow f_1 + f_2 \in \Phi$

and • $c f_1 \in \Phi$ for $c \in \mathbb{R}$

Ex: $\Phi = \{ax + b \mid a, b$

$f_1 = a_1 x + b_1$

$f_2 = a_2 x + b_2$ $\Rightarrow f_1 + f_2 = (a_1$

$\in \mathbb{R})$

$+a_2)x + (b_1, b_2)$

Why linear function spaces?

If they have small representation

i.e. $(a,b) \in \mathbb{R}$

then they are "efficient" representation of functions

Key question, given f, how to find f^* or $\phi \in \Phi$ to represent f or approx. f?

- Ideally, we want the "best" or closest representation of f!

Given f, find $\phi \in \Phi$ s.t. ϕ is as close to f as possible

The best approximation problem.

Aside: Note the similarity to finding best $\hat{x} \in \mathbb{N}$ to approx $x \in \mathbb{R}$

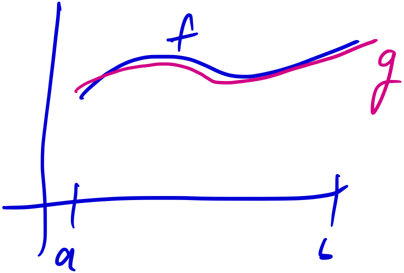
Function Norms

Is ϕ_1 or ϕ_2 a better approx to f?

⇒ We need a distance between functions!

A simple idea for distance

$\Rightarrow$ Look at it!



How close are f, g on $[a,b]$?

To quantify:

Let $\{x_i\}$ be a set of n (large) points in $[a,b]$

Compare f, g on these points!

E.g.

$$\underline{u} = \begin{bmatrix} f(x_1) \\ f(x_2) \\ \vdots \\ f(x_n) \end{bmatrix} \qquad \underline{v} = \begin{bmatrix} g(x_1) \\ \vdots \\ g(x_n) \end{bmatrix}$$

$\|\underline{u}-\underline{v}\|_2, \|\underline{u}-\underline{v}\|_1, \|\underline{u}-\underline{v}\|_\infty$.

Suppose we look @

$$\|\underline{u}-\underline{v}\|_\infty =$$

$$\max_{x_i} |f(x_i) - g(x_i)|$$

If $n \to \infty$,

$$\lim_{n\to\infty} \|u_n - v_n\|_\infty = \max_{a \le x \le b} |f(x) - g(x)|$$

Defn $\|f\|_\infty = \max_{a \le x \le b} |f(x)|$

$$\Rightarrow \|f-g\|_\infty = \max_{a \le x \le b} |f(x) - g(x)|$$

$$\|f-g\|_2 = \sqrt{\int_a^b |f(x) - g(x)|^2 \, dx}$$

$$\|f-g\|_1 = \int_a^b |f(x) - g(x)| \, dx$$

$$\downarrow$$

$$\|f\|_1 = \int_a^b |f(x)| \, dx$$

$$\|f\|_2 = \left(\int_a^b |f(x)| \, dx \right)^{1/2}$$

Example: Is x a good approx to x^2 around 1?

"around 1" $\Rightarrow \left[\overset{\overset{a}{\downarrow}}{1-h}, \overset{\overset{b}{\downarrow}}{1+h}\right]$

$$\left(\int_{1-h}^{1+h} (x^2-x)^2 \, dx\right)^{1/2} =$$

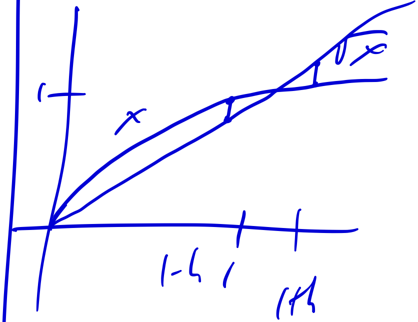
.. Wolfram Alpha ..

$$= \sqrt{\frac{2}{5}h^5 + \frac{2}{3}h^3}$$

If $h = .01 \Rightarrow 0.0008$

Yes!

Is $\sqrt{x}$ close to x near 1?



Soln

In the ∞-norm:

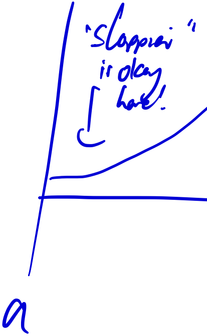
$$\|\sqrt{x} - x\|_\infty = \max_{a \le x \le b} |\sqrt{x} - x|$$

$$= \max\left(\begin{matrix} \sqrt{1-h} - (1-h) \\ 1+h - \sqrt{1+h} \end{matrix} \right)$$

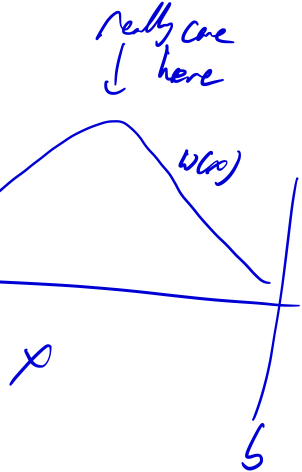
$\nearrow$

Key insight.

A concern:



$w(x) \ge 0$



is called a weight (function)

$$\|f\|_{1,w} = \int_a^b |f(t)|\, w(t)\, dt$$

$$\|f\|_{2,w} = \left(\int_a^b |f(t)|^2\, w(t)\, dt \right)^{1/2}$$

$$\|f\|_{\infty,w} = \max_{a \le x \le b} |f(t)|\, w(t)$$

A generalization

Let $\lambda(t) = \begin{cases} 0 & t < a \\ \int_a^t w(t) & a \le t \le b \\ \int_a^b w(t) & t > b \end{cases}$

Then a formal defn is

$$\boxed{\int_{t\in\mathbb{R}} f(t)\, d\lambda(t) = \int_a^b f(t) w(t)\, dt.}$$

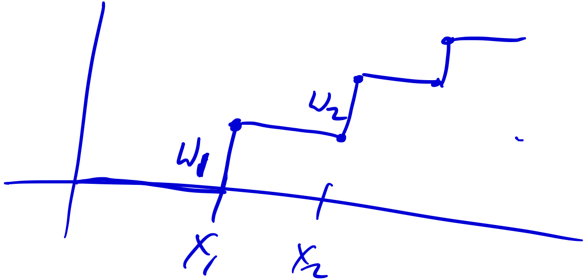
Stieltjes Integral.

If

$\lambda(t)$

$\int_{\mathbb{R}}$

$$\lambda(t) = \begin{cases} 0 & t < x_1 \\ \sum_{i=1}^{k} w_i & x_k \le t \le x_{k+1} \\ \sum w_i & t \ge x_n \end{cases}$$



$$f(t)\,d\lambda(t) = \sum_{i=1}^{n} f(x_i)\,w_i$$

The best approx prob

find $\phi \in \Phi$ s.t.

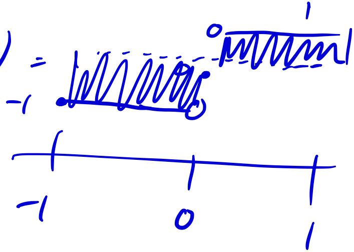
$$\| f - \phi \|$$

is as small as possible!

Question:
- Existence?
- Uniqueness?
- How to compute!

Example (Gautschi 2.1)

$f(x) =$



$c =$

$$\Phi = \{ c \mid c \in \mathbb{R} \}$$

$$\| f - \phi \|_1 = \int_{-1}^{1} | f(x) - c | \, dx$$

0

→ If $c \geq 0$, the

$$|f(x) - c| = \begin{cases} 1+c & x \in [-1,0) \\ c & x \in [0,0) \\ 1-c & x \in (0,1] \end{cases}$$

$$\Rightarrow \int_{-1}^{1} |f(x) - c| = (1+c) + (1-c)$$

$$= 2$$

$$\|f - \phi\|_2^2 =$$

$$\underset{x \in [-1,0)}{(1+c)^2}, \; \underset{x=0}{c^2}, \; \underset{x \in (0,1]}{(1-c)^2}$$

$$= 2 + 2c^2 \Rightarrow$$

$c = 0$ is best!

$\|f - \phi\|_\infty \Rightarrow c = 0.$
is best too!