Lecture 6

- Notes: Office Hour moved today ☹
  - 2-2:45 pm Today Only. Same Zoom as class.
  - HW 1 due Monday.

- A few loose Ends from Chapter 1.
  - → Condition # of $A\underline{x} = \underline{b}$
  - → Variance, Algebraic Eq.

→ Then Intro for Chapter 2

Chapter 1:

Solving $A\underline{x} = \underline{b}$

→ What is κ_f for a linear system?

⇒ What is f for a linear system?

$$x = f(\underline{b}) = A^{-1}\underline{b}$$

- Note A is "not" an input!
- Correspond to "worst" case of using Matrix exactly!

$$K_f(\underline{b}) = \frac{\|b\| \cdot \|J_f(b)\|}{\|f(b)\|}$$

What is Jacobian of $f(\underline{b}) = A^{-1}\underline{b}$?

What is $\frac{d}{db}(ab)$

a.

$f(\underline{b}) =$ "Matrix".

$$J_f(b) = \begin{bmatrix} \frac{\partial f_1}{\partial b_1} \\ \frac{\partial f_2}{\partial b_1} \\ \vdots \end{bmatrix}$$

$J_f = A^{-1}$

(Indep of b)

=

"vector"

$\partial f_i / \partial x_j$

$$K = \frac{\|b\| \cdot \|A^{-1}\|}{\|A^{-1}b\|}$$

We have $A^{-1}b = x$

$b = Ax$

$$K = \frac{\|Ax\|}{\|x\|} \cdot \|A^{-1}\|$$

$$K \le \|A\| \cdot \|A^{-1}\|$$

becau $\|A\| = \max_x \frac{\|Ax\|}{\|x\|}$

$K = \|A\| \cdot \|A$

is called the condition # of a matrix.

& if K is large, then solving

$$A\underline{x} = \underline{b}$$

is hard to get "ri answer.

$-1//$

$10^{12})$

ght"

Examples w/ Large Condition #s.

- Hilbert Matrix

$$H_n = \begin{bmatrix} 1 & 1/2 & \cdots & 1/n \\ 1/2 & 1/3 & 1/4 & \\ \vdots & 1/4 & 1/5 & \\ 1/n & & & \end{bmatrix}$$

$$[H_n]_{ij} = \frac{1}{i+j-1}$$

$$\kappa_2(H_{40}) = 10^{58}$$

Vandermonde

$x_i \in \mathbb{R}$

$$V_n = \begin{bmatrix} 1 & \cdots \\ x_1 & \\ x_1^2 & \\ \vdots & \\ x_1^{n-1} & \end{bmatrix}$$

$\kappa_\infty V_n \Rightarrow$

Matrix

$$\begin{bmatrix} 1 \\ x_n \\ \vdots \\ x_n^{n-1} \end{bmatrix}$$

n^{n+1}

Little Result

If you are analyzing "Alg Condition #'s"

Useful Lemma in Nick Higham's book

If $|\delta_i| \le u$, $\rho_i = \pm 1$ and $nu < 1$

Then $\prod_{i=1}^{n} (1+\delta_i)^{\rho_i} = 1+\theta_n$

$|\theta_n| \le \frac{nu}{1-nu}$

Van

(Sampl

Var

How

riance

6)

"the mean x"

$$r(\underline{X}) = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})^2$$

$$\bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i$$

to Compute Variance?

- Look @ all data, Compute mean $\bar{X}$
- Look @ all data (again) and Compute

$$\rightarrow Var(\underline{X}) = \frac{1}{n-1} \sum (X_i - \bar{X})^2 = \frac{1}{n-1} \sum_{i=1}^{n} [X_i^2 - 2\bar{X}X_i + \bar{X}^2]$$

$$-2\bar{X} \sum_{i=1}^{n} \frac{X_i}{n-1}$$

$\vdots$ (2nd Moment)

$$= \frac{1}{n-1} \sum X_i^2 - \frac{\bar{X}^2}{n(n-1)}$$

Your book shows how to do "this" algebra eq. → What is $K(y)$ w/r.t. $\underline{a}$?

$$p(x) = x^n + a_{n-1}x^{n-1} + \dots a_1 x + a_0$$

If y is a "root" of $p(x)$, i.e.

$$p(y) = 0$$

$$p'(y) \neq 0.$$

Chapter 2

THE QUESTION OF CHAPTER 1

How do we represent numbers on a computer and how does this representation impact what we might want to compute?

THE QUESTION OF CHAPTER 2

How do we represent mathematical functions on a computer?

The following may seem like a silly question, but what *is* a function f on a computer?

- Is it?

```
function f(x)
return sin(100x)
end
```

$f(x) = \sin(100x)$

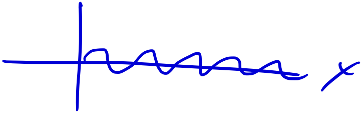
- Is it?

```
f(x) = 100x - 500000 x^3 / 3 + 250000000 x^5 / 3
```

- Is it?

```
pushq    %rax
imulq    100, %rdi, %rax
vcvtsi2sd    %rax, %xmm0, %xmm0
movabsq sin, %rax
callq    *%rax
popq     %rax
retq
```

- Is it a picture of f?



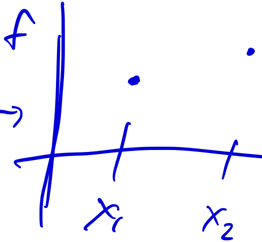
$x \in \mathbb{R}$

$f(x) : \mathbb{R} \to$

Map

fo

I can



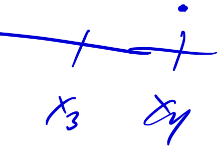
- infinite info

$\mathbb{R}$ a

An infinite infinite info

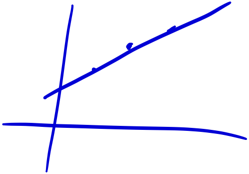
infinite infinite info!

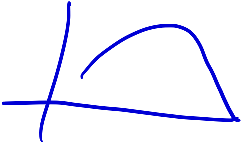
Eval f @ points!



w/ those Evals, I can

- Ignore x_4, and compute a linear approx!





Topic 1:

Computer Representation of functions

→ Linear function spaces (Eg. polynomials)

→ Splines

→ Trig-polys (Fourier analysis.)