Today: More Errors!

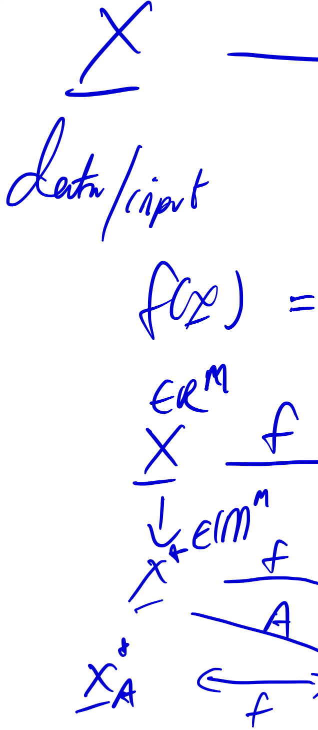
⊛ Fast approximation impacts.

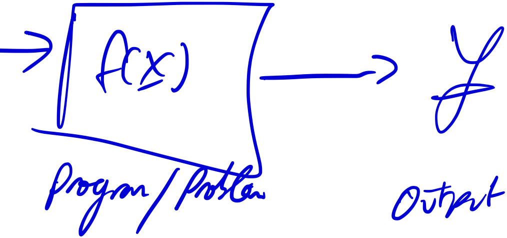
- The impact of the Algorithm

↳ Handouts online!

⊛ Exams: More details coming soon!

Computer Error





$A^{-1} \underline{x}$ "Solve a linear system"

$\Rightarrow f(\underline{x}) = y$

$\Rightarrow f(x^{\delta}) = y^{\delta}$

$\Rightarrow f_A(x^{\delta}) = y_A^{*} \in \mathcal{M}^n$

Last time

$f(x^{\delta})$ vs. $f(x)$

This time

$f(x_A^{*})$ vs. $f(x)$

ll

W

f

We

Our Mission!

$f(x_A^\delta) - f(x)\|$

le use

$f(x_A^\delta) - f(x) =$

$\nabla J(\underline{x})(\underline{x}_A^\delta - \underline{x})$

$\cancel{+ O(\|x_A^\delta - x\|^2)}$

We ignore these!

Machine ε / data roundoff.

ll, hopefully $\|x_A^\delta - x\| \approx O(u)$

[u for the notes!

$$\frac{\|f(x_A^\delta) - f(x)\|}{\|f(\underline{x})\|}$$

$$= \frac{\|\nabla J(\underline{x})(x_A^\delta - \underline{x}}{\|f(\underline{x})\|}$$

$$\le \frac{\|\nabla J(x)\| \cdot \|x_A^\delta - x\|}{\|f(x)\|}$$

Last time! $K_f(\underline{x})$

$$\le \boxed{\frac{\|J(x)\|\|\underline{x}\|}{\|f(\underline{x})\|}} \cdot$$

$) \|$

$$\frac{\|\underline{x}\|}{\|\underline{x}}$$

Now!

$$\frac{\|\underline{x}_A^* - \underline{x}\|}{\|\underline{x}\|}$$

Condition # of an Algorithm.

$$f_A : \mathbb{R}^m \to \mathbb{R}^m$$

Example $f(x_1, \ldots, x_m)$

$$= \prod_{i=1}^{m} x_i$$

Code/Algorithm

```
function myprod(x::Vector)
  p = x[1]
  for i=2:length(x)
    p = p*x[i]
  end
  return p
end
```

$f_A(\underline{x})$:

$p_1 = x_1$

$p_2 = fl(p_1$

$\vdots$

$p_m = fl(p_{m-1}$

Key assumption

$f_A(\underline{x}) =$

for some

(Backward

$\cdot X_2) = P_1 \otimes X_2$

$\cdot X_m) = P_{m-1} \otimes X_m$

$\vdots \quad ((\otimes))$

$f(X_A)$

e $X_A \in \mathbb{R}^m$.

Stability!)

Let's show this for

our Multiplication!

$$P_1 = X_1$$

$$P_2 = fl(P_1 X_2) = X_1 X_2 (1+\varepsilon_1)$$

$$P_3 = fl(P_2 X_3) = P_2 X_3 (1+\varepsilon_2)$$

$$= X_1 X_2 X_3 ((1+\varepsilon_1)(1+\varepsilon_2)$$

$$= X_1(1+\varepsilon_1) \cdot X_2(1+\varepsilon_2) \cdot X_3$$

$$\vdots$$

$$P_m = X_1(1+\varepsilon_1) \cdot X_2(1+\varepsilon_2) \cdot \ldots$$

$$X_{m-1}(1+\varepsilon_{m-1}) \cdot X_m$$

So x_A for Mult. Alg.

$$[x_1(1+\varepsilon_1), \; x_2(1+\varepsilon_2), \; \ldots \; x_{m-1}(1+\varepsilon_{m-1}), \; x_m]$$

$x_A^{(2)}$ could also be

$$\to [x_1, \; x_2(1+\varepsilon_1), \; x_3(1+\varepsilon_2), \; x_m(1+\varepsilon_{m-1})]$$

x_A is not unique!

Defn:

$$(\text{Cond } A)(\underline{x}) = \inf_{\underline{x}_A} \frac{\|\underline{x}_A - \underline{x}\|}{\|\underline{x}\|} \Big/ u$$

$$\le \text{any}_{\underline{x}_A} \frac{\|x_A - x\|}{\|x\|} \Big/ u$$

$$\left(\text{Cond}\ \begin{matrix}\text{Mult}\\ \text{Alg}\end{matrix}\right)(x) \le \frac{u \cdot \|x\|_\infty}{\|x\|_\infty} \Big/ u = 1$$

$$x_A - x =$$

$$\left[\varepsilon_1 x_1,\ \varepsilon_2 x_2,\ \ldots,\ \varepsilon_{m-1} x_{m-1},\ 0\right]$$

$$\|x_A - x\|_\infty \le u\,\|\underline{x}\|_\infty$$

n is dim of matrix!

$$\left(\text{Cond Alg}^{\text{GEPP}}_{(\backslash \text{ in Julia})}\right)(x) \le 2^n$$

Another Example

$$f(x) = \log x$$

Given Alg s.t.

Not yet a BW stability statement.

$$\boxed{f_A(x) = \log(x)(1+\varepsilon)}$$

when $|\varepsilon| \le 5u$.

$$\boxed{f_A(x) = \log(x^{(1+\varepsilon)})}$$

is backwards stable!

$$(\text{Cond } A)(x) = \frac{|x - x_A|}{|x|} = \frac{|x^{(1+\varepsilon)} - x|}{|x|}$$

$$= |x^{\varepsilon} - 1| = |e^{\varepsilon \log x} - 1|$$

$$\approx |\varepsilon \log x|$$

$$(\text{Cond } A)(x) \le 5 |\log x|$$

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Complete Error Analysis

$$\frac{\|f(x_A^\dagger) - f(\underline{x})\|}{\|f(x)\|} \le \kappa_f(\underline{x}) \frac{\|x_A^\dagger - x\|}{\|\underline{x}\|}$$

$$\|x_A^\dagger - x\| = \|x_A^\dagger - x^\dagger + x^\dagger - x\|$$

$$\le \|x_A^\dagger - x^\dagger\| + \|x^\dagger - x\|$$

$$\frac{\|x_A^\dagger - x^\dagger\|}{\|x\|} + \frac{\|x^\dagger - \underline{x}\|}{\|\underline{x}\|} \le \overset{\varepsilon}{\cancel{(1+\varepsilon)}}$$

Input as Machine #s.

$$\frac{\|x_A^\dagger - x^\dagger\|}{\|x\|} \cdot \frac{\|x^\dagger\|}{\|x^\dagger\|}$$

$$= \underbrace{\frac{\|x_A^\dagger - x^\dagger\|}{\|x^\dagger\|}}_{\text{This is cond } A\,!} \quad \underbrace{\frac{\|x^\dagger\|}{\|x\|}}_{\text{Assume } O(1+\varepsilon)}$$

$\kappa_A(x^\dagger)u$

$\frac{\|x^\dagger\|}{\|x\|}$ is $\le 1+\varepsilon$ for $\|x\|_\infty$.

$$\frac{\| f(x_A^+) - f(x) \|}{\| f(x) \|} = \kappa_f(x) \Big[(\text{Cond } A)(x^+)\, u \, C(f\epsilon) +$$

$u\epsilon$ is very small!

$$\approx \overbrace{\kappa_f(x)}^{\text{Math function!}} \Big[\overbrace{(\text{Cond } A)(x^+)\, u}^{\text{Alg}} \; \overset{\text{Input}}{\overset{\downarrow}{f\epsilon}} \Big]$$

(Book gets this a different way w/ different assumptions.)

ε)

Implications

- Good Algs have (Cond A)(x^*) Small
- Hard problems have $\kappa_f(x)$ Large!

i.e $\kappa_f(x) = 10^9$, you need ε, u small to get Small Error!

If $\kappa_f(x)$ is large, then you can't fix that with an Algorithm!

(Need Smaller u!)