Lecture 4

- See notes/handout on web page!
- Goals in Error Analysis
- Aside on Norms
- Condition # of a problem!

Logistics!

- Video HW due today!
- HW 1 posted soon (afternoon?)
- Office Hours
  - ↳ 1-2pm MC! Today, Thurs.
  - ↳ 3pm Sai Friday!

Same Zoom Link!

Your Goal: Solving a problem P.

$$\underline{x} \longrightarrow \boxed{f} \longrightarrow y$$

P / f : Subtract

$f(x, y) = x - y$

$f(x) = \log(x)$

$f(x, y) = xy$

$f(\underline{x}) = A^{-1} \underline{x}$

$f(\underline{x})$ = roots of a poly w/ coeffs $\underline{x}$

The Setup

$\underline{x} \in \mathbb{R}^m$

Put $\underline{x}$ on machine $\downarrow$

• $\underline{x}^* \in \mathbb{M}^n$

• $\underline{x}_A^* \in \mathbb{R}$

• Backwards S

$f_A(x^*) =$

"y_A^*"

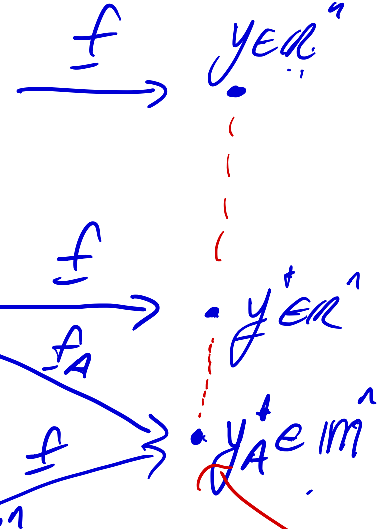


table Algs

$f(x_A^+)$ for some x_A^+ "not too far away"

The goal

Understand

y vs. y_A^+

Quantatively

$$\frac{\|y - y_A^+\|}{\|y\|}$$

How do we do this?

- Connect $\boxed{\text{the dots!}}$

2 major analyses

→ y vs. y^+ — Just Calc — "Easy"

and y_A^+ vs. y^* — Algs, floats, loops, etc. → Next Class.

⊕ Depends on the Condition # of the Prob P/ function $\underline{f}$.

Norms : We need

- vector + matrix norms to quantify these quantities

$\underline{x} \in \mathbb{R}^m$

Notation vectors underlined

p-Norms

$$\|\underline{x}\|_p = \left(\sum_{i=1}^{m} |x$$

"Euclidean" norm

$$\|x\|_2 = \sqrt{\sum_{i=1}^{m} |x_i|^2}$$

$$\|x\|_1 = \sum_{i=1}^{m} |x_i|$$

are

!

$x_i|^p)^{1/p}$

$(p=2)$

$\|x\|_\infty = \max_i |x_i|$

Any vector norm satisfy
(p= anything ...)

1. $\|x\| \ge 0$
2. if $\|x\|=0$, then $x=0$.
3. $\|x+y\| \le \|x\|+\|y\|$
4. $\|\alpha x\| = |\alpha|\,\|x\|$

A matrix is a coordinate represent of a linear transform from $\mathbb{R}^n$ to $\mathbb{R}^m$.

$y = A\underline{x}$ is the transform from x to y.

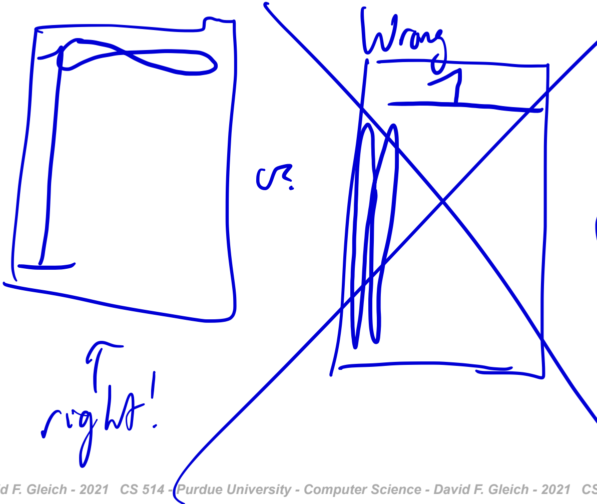
Induced Matrix Norms

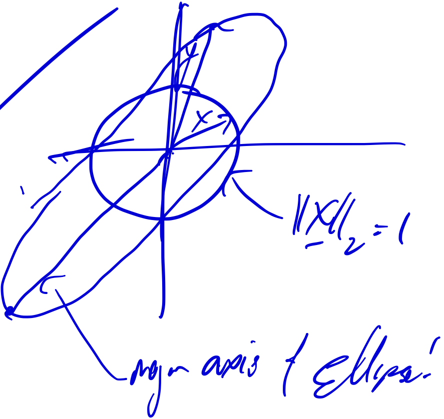
$$\|A\|_p = \max_{\underline{x}\neq 0} \frac{\|A\underline{x}\|_p}{\|\underline{x}\|_p}$$

$$\|A\|_1 = \max_j \sum_{i=1}^{m} |A_{ij}|$$

$$\|A\|_2 = \text{max sing. value} = \sqrt{\text{max eigenval of } A^T A}$$

$$\|A\|_\infty = \max_i \sum_{j=1}^{n} |A_{ij}|$$





1. $\|A\| \ge 0$
2. $\|A\| = 0 \Rightarrow A = 0$
3. $\|A+B\| \le \|A\| + \|B\|$
4. $\|\alpha A\| = |\alpha| \, \|A\|$
5. $\|AB\| \le \|A\| \cdot \|B\|$

Induced Norm

Done w/ Norms

Back to y vs. $\hat{y}$

$\rightarrow$ Condition #

Start Simple: Scalar value f:

$f : \mathbb{R} \rightarrow \mathbb{R}$

Eg. $f(x) = 1/\log(x)$

Goal $y = f(x)$

$\hat{y} = f(\hat{x})$

$\Delta y = \hat{y} - y$ $\quad \Delta x = \hat{x} - x$

Goals:

$$\frac{\Delta y}{|y|} \quad \text{vs.} \quad \frac{\Delta x}{|x|}$$

$$\frac{\Delta y}{|y|} \quad \overset{??}{\underset{??}{\boxed{}}} \quad \frac{\Delta x}{|x|}$$

If Δx "Small Enough"

$$\boxed{\Delta y \approx f'(x)\,\Delta x}$$

Taylor thm!

$$\frac{\Delta y}{|y|} \approx \frac{|x| f'(x)}{|y|} \frac{\Delta x}{|x|}$$

$$\left|\frac{\Delta y}{y}\right| \approx \boxed{\left|\frac{x f'(x)}{f(x)}\right|} \left|\frac{\Delta x}{x}\right|$$

Called the Condition # of f.

Gautschi writes

$$(\text{cond } f)(x) = \left| \frac{x f'(x)}{f(x)} \right|$$

$$\downarrow$$

$$\kappa_f(x)$$ (others/me)

"Kappa"

Todo: you → Compute

$$\left(\text{cond } \frac{1}{\log(x)}\right)(x)$$

Multivariate f

$$f : \mathbb{R}^m \to \mathbb{R}^n$$

2 types of condition #s

☹ or ☺

Both use the Jacobian

$$J(\underline{x}) = \begin{bmatrix} \partial f_1/\partial x_1 & \cdots & \partial f_1/\partial x_m \\ \partial f_2 \partial x_1 & & \vdots \\ \vdots & & \\ \partial f_n/\partial x_1 & & \end{bmatrix}$$

$n \times m$ matrix

Condition # 1

$$\gamma_{ij}(\underline{x}) = \left| \frac{x_j \, J_{ij}(x)}{f_i(\underline{x})} \right|$$

$$\Gamma(\underline{x}) = \left[\gamma_{ij}(x) \right]_{\substack{i=1 \text{ to } n \\ j=1 \text{ to } m}}$$

$\|\Gamma\|$ is a matrix norm.

$$f(x,y) = xy$$

$$J(x,y) = \begin{bmatrix} y & x \end{bmatrix}$$

$$\Gamma(x,y) = \begin{bmatrix} \frac{|xy|}{|xy|} & \frac{|xy|}{|xy|} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \end{bmatrix}$$

$$\|\Gamma\|_\infty = 2$$

Condition # 2

(More like

(

$\|\Delta y\| \approx$

Multivar Taylor

$$f(x + \Delta x) \approx$$

$$\frac{\|\Delta y\|}{\|y\|} \le$$

→ Sloppy!

Taylor)

$\|J(x)\Delta x\|$

$= \underline{f}(x) + J(x)\Delta \underline{x}$

↓

$$\boxed{\frac{\|x\| \cdot \|J_f(x)\|}{\|f(x)\|} \cdot \frac{\|\Delta x\|}{\|x\|}}$$

$\|f(x)\| = \|y\|$

$$(\text{Cond } \underline{f})(\underline{x}) = \frac{\|J(x)\| \cdot \|x\|}{\|f(\underline{x})\|}$$

Lets use ∞-norm.

$\|J(x)\|_\infty = |x| + |y|$

$$(\text{Cond } (xy))(\underline{x}) = \frac{(|x|+|y|)\max}{\|x\|}$$

is "worse" than previous!

This analysis is sloppier!

$\| J(x) \Delta x \|$ vs.

$\le \| J(x) \| \cdot \| \Delta x \|$

Todo: $f(x,y) = x - y$

$C(x_1, y_1)$