Lecture 25

Forward or Explicit Euler
(or First order extrapolation)
(or Forward extrapolation)
or Forward difference Scheme
Backward or Implicit Euler
(or Implicit first order extrapolation?)
or Implicit backward differencing.

A general analysis of one-step methods and the ideas of consistency and truncation error

Given f - a function of t(t,y)
and a grid of points $t1 < t2 < \ldots < tN$
and y0 then

First order extrapolation
Let y^(t0) = y(t0) # initial condition
For j=0 to N-1
$y^{(j+1)} = y^{(j)} + (t_{(j+1)}-t_{(j)})\, f(t_{(j)}, y^{(j)})$

Julia:
function first_order_extrapolation(f, t0, y0::Vector)
y = y0
for i=1:length(t0)-1
y = y + (t0[i+1]-t0[i])*f(t0[i], y)
end
end # todo, do something with y, store each vector?

function first_order_extrapolation(f, t0::Real, dt::Real, N::Int, y0::Vector)
y = y0
t = t0
for i=1:N
y .+= dt*f(t, y)
t += dt
end
end # todo, do something with y, store each vector?

$\frac{dy}{dt}(t_i) = \frac{y(t_{i+1}) - y(t_i)}{t_{i+1} - t_i} = f(t_i, y(t_i))$

$\Rightarrow y(t_{i+1}) = y(t_i) + (t_{i+1} - t_i)\, f(t_i, y(t_i))$

Given $\frac{dy}{dt} = f(t, y)$, $y(t_0) = y_0$

Approx $y(t)$ at $t_1 < t_2 < \ldots < t_N$

$t_i = dt \cdot i$ (uniform grid)

Let see on $\frac{dy}{dt} = Ay$

Let look at

$y_0 = y(t_0)$
$y_1 = y(t_1)$
$y_2 = y(t_2)$
$\vdots$
$y_N = y(t_N)$

Assume uniform time grid of step dt

$y_1 = y_0 + h\, A y_0 \Leftrightarrow y_1 - (I + hA) y_0 = 0$

$y_2 = y_1 + h\, A y_1 \Leftrightarrow y_2 - (I + hA) y_1 = 0$

A great big System

$$\begin{bmatrix} I & & & \\ -(I+hA) & I & & \\ & -(I+hA) & I & \\ & & & -(I+hA) \; I \end{bmatrix} \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \\ y_N \end{bmatrix} = \begin{bmatrix} y_0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Soln:

$y_1 = (I+hA) y_0$

$y_2 = (I+hA) y_1 = (I+hA)^2 y_0$

$\vdots$

$y_N = (I+hA)^N y_0$

Very handy for Analysis!

Backward Soln

- $f'(x) \approx \frac{f(x+h) - f(x)}{h}$ (forward diff.) a)
- $f'(x) \approx \frac{f(x) - f(x-h)}{h}$ (backward diff.)
- $\frac{dy}{dt}(t_1) = \frac{y(t_1) - y(t_0)}{t_1 - t_0} = f(t_1, y_1)$

$\frac{y_1 - y_0}{t_1 - t_0} = f(t_1, y_1) \Leftrightarrow$ dont know y_1! we only know y_0!

Soln: Use a System of non-lin eq!

Let $z = y_1$, then find z s.t.

$f(t_1, z)(t_1 - t_0) + y_0 - z = 0$

So Each step is a non-linear Eq Solve!

Backward Soln on $\frac{dy}{dt} = Ay$

$f(t_{1,2}) = A_2$

$\Leftrightarrow$

$A_2 \frac{(t_1 - t_0)}{h} + y_0 - z = 0$

$h A_2 - z + y_0 = 0$

$(I - hA) z = y_0$

$z = y_1 = (I - hA)^{-1} y_0$

$\vdots$

$y_N = (I - hA)^{-N} y_0$

Forward Soln $(I + hA)^N$

Backward Soln $(I - hA)^{-N}$

Last point:

for Grid Approx, we usually have ...

Spse

$$\begin{bmatrix} I & & & \\ -(I+hA) & I & & \\ & -(I+hA) & I & \\ & & & -(I+hA) \; I \end{bmatrix} \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \\ y_N \end{bmatrix} = \begin{bmatrix} y_0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

for poly approx of $y(t)$, we get a fully Coupled System!

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Notes online too

THE PROBLEM

We are considering numerical methods for the initial value problem:

$$\frac{dy}{dt} = f(t, y) \qquad y(0) = y_0, t \in [0, T]. \tag{1}$$

where f is continuous and outputs an $\mathbb{R}^d$ vector.

EXISTENCE AND UNIQUENESS

THEOREM 1 (Gander, p.20). Theorem 6.1.3: *Assume that $f(t, y)$ is continuous in the first variable (t) in the range $[0, T]$ and with respect to the second variable $\{y\}$, we satisfy a uniform Lipschitz condition:*

$$\|f(t, y_1) - f(t, y_2)\| \leq L\|y_1 - y_2\|, \qquad t \in [0, T], y_1, y_2 \in \mathbb{R}^d.$$

(The norm can be arbitrary) Then the problem (1) has a unique solution $y(t), 0 \leq t \leq T$ for arbitrary y_0 and the solution depends continuously on y_0.

The only hard part about this statement is the Lipschitz condition. This is called Lipschitz continuity too.[1]

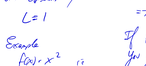
A Sufficient Condition for this problem (in theory) is a bounded derivative on f.

· Example

$\sin(x), \cos(x)$ are Lipschitz w/ $L = 1$

· Example

$f(x) = x^2$ is not Lipschitz for any L.



Example:

If $f'(x)$ bounded then $|f'(x)| \leq L$

$\Rightarrow$ Lipschitz.

If $f: \mathbb{R}^d \to \mathbb{R}^d$, you get the same of operator norm of Jacobian.

Why Lipschitz?

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EXAMPLE 1. *That page has a great example, which is*

$$dy/dt = y^2, \quad y(0) = 1, \qquad \text{then} \qquad \boxed{y(t) = 1/(1-t)}.$$

This function does not exist at $t = 1$.

(*) Actually T.

Locally Lipschitz is Lipschitz on an interval!

Eg. x^2 is locally Lipschitz on any interval.

$f(x) = 2x \leq$ Max $2x$ on int

$\Rightarrow$ L-max Lipschitz.

Local Lipschitz $\Rightarrow$ finite time Existence!

Eg.

$$\frac{dy}{dt} = y^2$$

Focus on Grid Above

The methods we will consider in this class are all grid-approximations of the function $y(t)$. That is, we consider

$$y(t_0) = y(0), y(t_1), \ldots, y(t_N) \qquad t_0 = T$$

and usually uniformly spaced grids where $t_i = ih$ for some $h = T/N$.

Let $y_i = y(t_i)$ for convenience.

We often look at $y_i \to y_{i+1}$

So we will use the Shorthand

$$\boxed{y \to y^+}$$

clearer to read!

$y_i, y_{i+1}, h, f(t, y)$

$y^+, y, h, f(t, y)$

Focus on Explicit methods

Eg. No Implicit Solvers!

A general Setup

$$y^+ = y + h\boxed{\Phi(t, y, h)}$$

Abstract approach

Next y Current y diff eq info. step size. Use the Grid, h!

Another method

$$\boxed{f(t+h, y + h f(t, y))}$$

y^+ = Forward Euler.

$f(t, y)$

Current Slope

t $t+h$

Heun uses $\Phi(t, y, h) = \frac{1}{2} f(t, y) + \frac{1}{2} f(t+h, y + h f(t, y))$

FE point.

FF uses $\Phi(t, y, h) = f(t, y)$

Another Idea: Midpoint Solve $f(t + \frac{h}{2}, y + \frac{h}{2} f(t, y))$ use for slope

$$\Phi(t, y, h) = f(t + \tfrac{1}{2}, y + \tfrac{h}{2} f(t, y))$$

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