Lecture 22

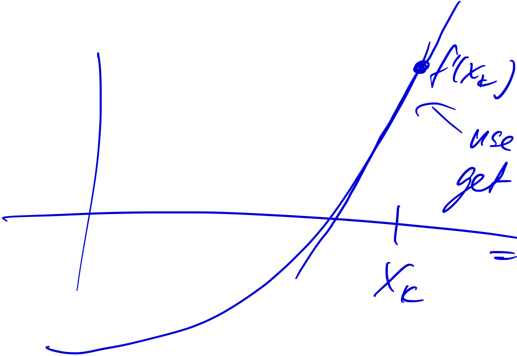
⊛ Exam Thursday!

- More on Newton + Fixed pt. Methods
- Long tangent on fixed point Convergence

  why → Newton goes faster! (quadratic Conv.)
- HW 4
- Vaccines!
- Exam Review

Recall Newton

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$



$\mathbb{R} \xrightarrow[\cdot]{\cdot} \mathbb{R}$

deriv to tangent line.

→ solve for zero!

Q: Will it converge?

How fast??

Example:

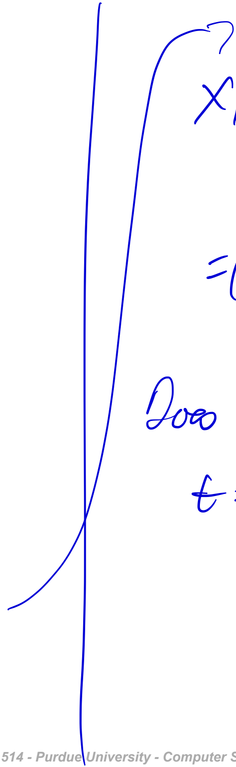
Goal: Compute

$c^{1/t}$

$f(x) = x^t - c = 0$

$\iff x = c^{1/t}$

$(t = 1/5)$



$x_{k+1} =$

$= (1 - \frac{1}{t}$

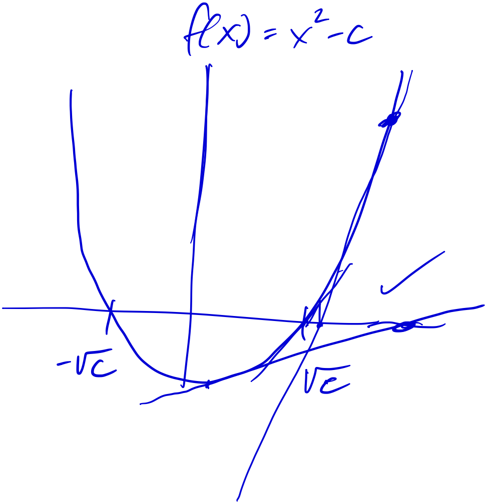
Does this

$t = 2$

$$x_k - \frac{x_k^t - c}{t x_k^{t-1}}$$

$$-) \; x_k + \left(\frac{c}{t}\right) x_k^{(1-t)}$$

Converge ??



If $x_0 = 0$ - $f'(x_0) = 0 \Rightarrow$ divide by zero ☹

As long as $0 < x_0 < \infty \Rightarrow$

Converge.

$t=2 \Rightarrow$

$$x_{k+1} = \frac{1}{2}x_k + \frac{1}{2}c\,x_k^{-1}$$

We can study further (if we want!)

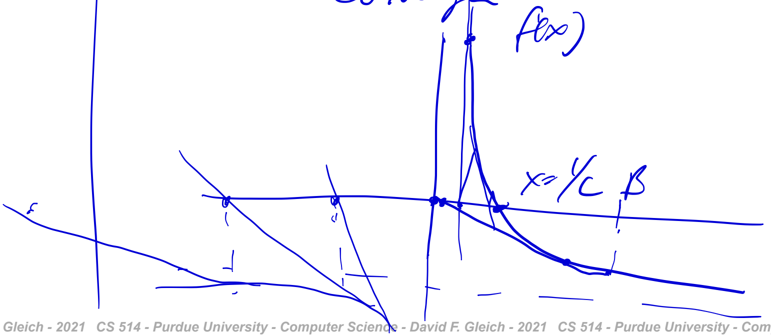
Ex 2:

$$f(x) = \frac{1}{x} - c \iff x = \frac{1}{c}$$

$$x_{k+1} = 2x_k - c\,x_k^2$$

Idea: division w/o division!

Look at Convergence



If $0 < x_0 < 1/c$

=> Converge...

If $x_0 < 0$,

=> Not Converge ☹

If $x_0 > \beta$, then

$x_1 < 0$ =>

Not Converge ☹.

$\beta \Leftrightarrow x_1 = 0 \Leftrightarrow$

$2\beta - c\beta^2 = 0$

$\beta = 2/c$.

for

$0 < x_0 < \beta = 2/c$

we get Convergence!

How fast does this converge?

→ Newton →

Follow the tangent!

We will take a tangent into fixed pt. method!

- We will show that Newton (close enough) goes quadratically fast to the root.

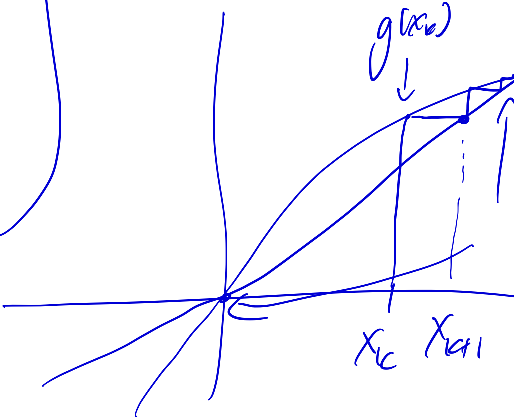
Goal: $|x_{k+1} - \overset{\text{root}}{\alpha}| \sim |x_k$

If $f(x) = 0$ can be written as $g(x) - x = 0$ for some function g. then finding a root of f is like finding a fixed point of g

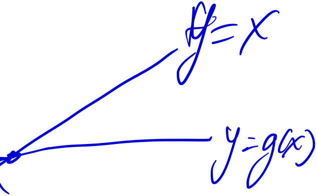
$$g(x) = x \iff \text{root of } f.$$

$g(x) = x$ suggests the iteration

$$x_{k+1} = g(x_k).$$



$-\alpha)^2$



fixed pt.

Thm: If

$$\boxed{|g'(x)| \le \lambda < 1}$$

Then if also

$$|x_0 - g(x_0)| \le (1-\lambda)\rho$$

for some ρ.

⊛ There is a Lipschitz version of this too that is more general!

$\rightarrow$ We

x_k

the u

w/in

$\boxed{x_0} - \rho$,

get

$\to \alpha$

nique root

$x_0 + \rho \Big].$

Why unique?

By Contradiction:

If α, β are 2 distinct ~~roots~~ fixed pts

$|\alpha - \beta| = |g(\alpha) - g(\beta)|$

$= |g'(\eta)(\alpha - \beta)|$

for $\eta \in (\alpha, \beta)$ by mean value thm.

$\le \lambda |\alpha - \beta|$

$< |\alpha - \beta| \Rightarrow$

Contradiction.

How

$|x_k$

$|g($

λ

$\lambda |$

$\lambda^2 |$

$\cdots \lambda$

and λ

linear

fast?

fixed pt.

$n - \alpha| =$

$x_k) - g(\alpha)| \le$

$| x_k - \alpha| \le$

$g(x_{k-1}) - g(\alpha)| \le$

$x_{k-1} - \alpha| \le$

$^{k} | x_0 - \alpha|$

$< 1 \Rightarrow$

Convergen.

We can do better analysis!

$|\alpha - x_{k+1}| =$

$|g(\alpha) - g(x_k)| =$

$|g'(\eta)|\,|\alpha - x_k|$

for $\eta \in [\alpha, x_k]$

But $x_k \to \alpha$, so

$|g'(\eta)| \to |g'(\alpha)|$

So the asymptotic rate of convergence is given by $g'(\alpha)$.

Why this tangent!?!

* what if $g'(\alpha) = 0$?

we go faster & faster!

Sorry!

App Crashed!

Taylor around

$$g(x) = g(\alpha)$$

$$+$$

If $x = x_k$,

$$g(\underset{\substack{\| \\ x_{k+1}}}{x_k}) = \underset{\alpha}{g}$$

$$|x_{k+1} - \alpha|$$

α, fixed poin

$+ g'(\alpha)(x-\alpha)$

$g''(\eta)\frac{(x-\alpha)^2}{2} \quad \eta \in [x, \alpha]$

then if $g'(\alpha) = 0$

$(\alpha) + \overset{=0}{g'(\alpha)}(x-\alpha) + g''(\eta)\frac{(x_t-\alpha)^2}{2}$

$||$

$-$

$\Rightarrow$

$= \frac{|g''(\eta)|}{2} \cdot |x_t - \alpha|^2 \Rightarrow$ good hint about quadratic convergence.

More analysis

$$|x_{k+1} - \alpha| \le M^{(-1+2^k)} \underbrace{|x_0 - \alpha|^{2^k}}$$

Where $|g''(x)| \le 2M$

$M|x_0 - \alpha| < 1$

Where is

$x = g(x)$

Where $f(\alpha$

$g(\alpha$

So α is a

g

and α is a

⇒ use ϕ to

ewton?

$+ \phi(x) f(x)$

$) = 0$

$= \alpha.$

fixed point of

root of f.

make $g'(\alpha) = 0$.

$$g'(x) = 1 - \phi(x) f'(x) - \phi'(x) f(x)$$

$g'(\alpha) = 0$ w/ $f(\alpha) = 0$

$$\Rightarrow \phi(\alpha) = \frac{1}{f'(\alpha)}$$

$g(x) = x - \dfrac{f(x)}{f'(x)} \Rightarrow x_{k+1} = x_k$

$\boxed{\text{Newton's iteration as a fixed pt.}}$

$$g'(x) = \frac{f(x) f''(x)}{[f'(x)]^2}$$

$$g''(\alpha) = \frac{f''(\alpha)}{f'(\alpha)}$$

$$- \frac{f(x_k)}{f'(x_k)}$$

More analysis gives

$$\lim_{k \to \infty} \frac{|x_{k+1} - \alpha|}{|x_k - \alpha|^2} = \frac{1}{2} g''(\alpha) = \frac{1}{2} \frac{|f''(\alpha)|}{|f'(\alpha)|}$$

If $f'(\alpha)=0$ for Newton! we have a problem!

Ex: Newton to minim

minimize $\theta(x)=x^4$

$\iff$ find root of

$f(x)=4x^3$

$f'(x)=12x^2=0$ $\rightarrow$

Slower than quadratic!

$$x_{k+1}=x_k-\frac{4x_k^3}{12x_k^2}$$

$$=x_k-\frac{1}{3}x_k$$

$$=\left(1-\frac{1}{3}\right)x_k$$

$$=\frac{2}{3}x_k$$

$\Rightarrow$ linear converg

$\lim_{k\to\infty} x_k=0$, rate = linear conv

$2/3$.

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