Lecture 20

Goal:

$f: \mathbb{R} \to \mathbb{R}$

find α s.t.

$$f(\alpha) = 0$$

Method of Bisection

Given $[a_0, b_0]$

where $f(a_0) f(b_0) < 0$

(i.e. alternate signs)

Assume f is continuous

Iterate:

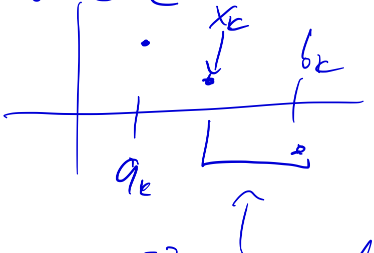
$$x_k = \frac{a_k + b_k}{2}$$

If $\text{sign}(f(x_k)) = \text{sign}(f(a_k))$

→ then repeat on

$[a_{k+1}, b_{k+1}] = [x_k, b_k]$

i.e. Case



⇒ root in the region

else: Repeat on

$[a_{k+1}, b_{k+1}] = [a_k, x_k]$

else: $f(x_k) = 0$ → lucky! done! ☺

* Floating pt. Implementations are tricky! See the Ref. on the web-page.

When to stop?

- $[a_k, b_k]$ contain exactly 2 float pt. #s: a_k, b_k.

Pick based on smaller of $|f(a_k)|$ vs $|f(b_k)|$

Alt: think about use case! Is over-approx. better? under-approx?

Earlier stopping is possible: when $|b_k - a_k| \le \text{tol}.$.

Is bisection fast?

- $|a_k - b_k| = \frac{1}{2^k} |a_0 - b_0|$

If $k \ge 64$, we reduce interval down to floating pt. (64 bits, pick "one-bit" each time...)

Realistically, $k = 52$ should be right if you get the exponent set

Poll:

Yes — 8/18

No — 3/18

Maybe — 6/18

If f is fast enough. then yes!

Very predictable Runtime!

How to Study?

§4.2 - Rates of Convergence!

Setup:

$$\lim_{n\to\infty} x_n = \alpha$$

$$f(\alpha) = 0.$$

How fast does x_n

where

Converge?

Defn: linear convergence at rate γ

$\Leftrightarrow$

$$|x_n - \alpha| \le \gamma^n C$$

for some const. C and $\gamma < 1$

Bisection satisfies this defn!

$\gamma = \frac{1}{2}$, $C = |a_0 - b_0|$

A mo

linear

$\lim_{n \to \infty}$

we also

Linear Con

Sometimes

Convergence

e general defn of Convergence.

$$\frac{|x_{n+1} - \alpha|}{|x_n - \alpha|} = \gamma$$

$(\gamma < 1)$

Say linear convergence.

vergence MAY
be called Exponential
!

Aside why Exponential?

$\gamma^n \to 0$

"Exponentially" fast!

Also called geometric Convergence.

Other Speeds:

Convergence of order

$$\lim \frac{|x_{n+1} - \alpha|}{|x_n - \alpha|^p}$$

$(C > 0)$

(and $C < 1$ if $p =$

Linear: $p = 1$

Quadratic $p = 2$

Cubic $p = 3$

p if

= C

/)

Example of Cubic Convergence

$$\boxed{|x_k - \alpha| = 0.1}$$

$$\boxed{|x_k - \alpha|^3 = 0.001}$$

So we need

$$\boxed{|x_{k+1} - \alpha| \approx C \cdot 0.001}$$

At 2 steps:

$$|x_{k+2} - \alpha| \approx C^4 \cdot 10^{-9}$$

Cubic Converge →

triple # of

Correct digits

at Each Step!

$\Downarrow$

4-5 Steps for

full floating

pt. accuracy!

Algebraic (Example)

$$|x_n - \alpha| \le \frac{1}{n^p} C$$

$$\frac{1}{n^p} \quad \text{vs.} \quad \gamma^n$$

algebraic / sublinear

linear or exponential

$\Downarrow$

$$\lim \frac{|x_{n+1} - \alpha|}{|x_n - \alpha|} = 1 \Rightarrow \text{sublinear.}$$

If $\lim_{n \to \infty} \frac{|x_{n+1} - \alpha|}{|x_n - \alpha|} =$

We call it super-linea

O

Examples of Algorithms

- Monte Carlo Integration
  Algebraic $\frac{1}{n^{1/2}}$ rate.
- Trapezoid Rule
  Algebraic $\frac{1}{n^2}$ rate.
- Linear - bisection
  - false position
- Super linear - Secant method.

- Quadratic -
  Method.
- Cubic - λ-

⋮

- Nonic approx
  9th order

1 → 9

lots!

Newton's

vals.

to $\frac{1}{\pi}$.

$\to 81 \to$

Example

Show

$x_n = \frac{1}{2^{2^n}}$ is quadratic to 0.

(i.e $\alpha = 0$.

Why?

$$x_{n+1} = \frac{1}{2^{2^{n+1}}}$$

$$x_n = \frac{1}{2^{2^n}}$$

x_n

$=$

$=$

^{2}

$$\left.\begin{aligned}
&= \left(\frac{1}{2^{2^n}}\right)^2 \\
&= \frac{1}{2^{2^n}} \cdot \frac{1}{2^{2^n}} \\
&\frac{1}{2^{2^n+2^n}} = \\
&\frac{1}{2^{2\cdot 2^n}} = \frac{1}{2^{2^{n+1}}}
\end{aligned}\right\} \rightarrow \frac{|x_{n+1} - 0|}{|x_n - 0|^2} = 1$$

h

What is cubic?

$$\frac{1}{2^{3^n}}$$ is cubic!

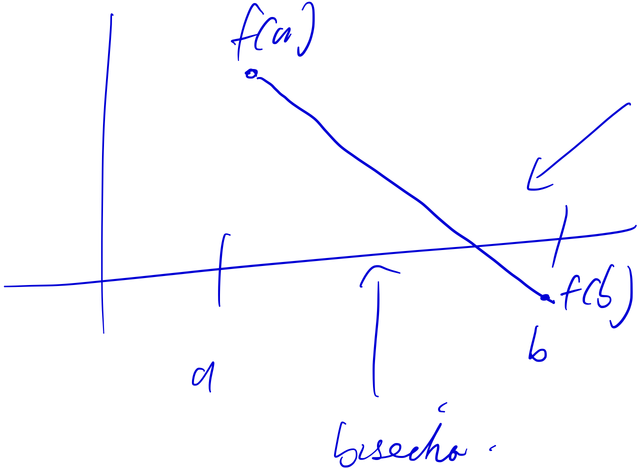
hy?

$$x_n = \frac{1}{2^{3^n}} \Rightarrow x_n^3 = \frac{1}{2^{3^n}} \cdot \frac{1}{2^{3^n}} \cdot \frac{1}{2^{3^n}}$$

$$x_{n+1} = \frac{1}{2^{3^{n+1}}}$$

$$= \frac{1}{2^{(3^n+3^n+3^n)}} = \frac{1}{2^{(3\cdot 3^n)}} = \frac{1}{2^{3^{n+1}}}$$

False Position! vs Bisection



pick 0 of the line connecting f(a), f(b)

=> False Position!

Alg: Given a_0, b_0
that contain a root.

Assume f is continuous.

Let $$x_k = a_k - \frac{a_k - b_k}{f(a_k) - f(b_k)} f(a_k)$$

(How to derive? Write
Interpolate poly in Newton form
Solve for zero!)

If $f(x_k) \cdot f(a_k)$
(then they ha
sign!)

then set a_{k+1}

else set a_{k+1}, b

⊕

> 0

ve the same

$b_{k+1} = x_k, b_k$

$_{k+1} = a_k, x_k.$

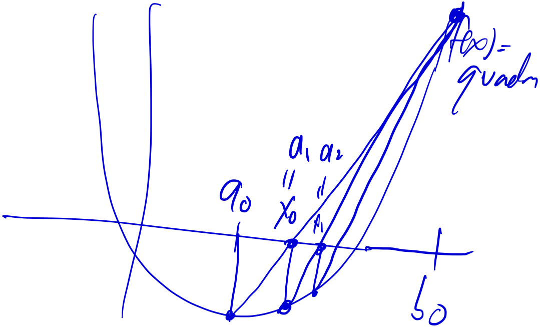
Let's do some analysis

Assume f is convex.

$f(a_0) < 0, \ f(b_0) > 0$

Only 1 zero in $[a_0, b_0]$

Eg.



Convex $\Rightarrow$ will only change a at Each step.

$$x_{k+1} - \alpha = x_k - \alpha - \frac{x_k - b}{f(x_k) - f(b)}$$

$a_{k+1} = x_k$

$x_k < \alpha$ (true root(s))

$$\lim_{k \to \infty} x_k = \alpha.$$

$x_k - \alpha$

$$\frac{x_{k+1} - \alpha}{x_k + \alpha} = 1 - \frac{x_k - b}{f(x_k) - f(b)}$$

$$\lim_{k \to \infty} \Rightarrow x_k \to \alpha \quad = 1 -$$

$$\left[f(x_k) - f(\alpha) \right)$$

$$\frac{f(x_k) - f(\alpha)}{x_k - \alpha} \approx f'(\alpha)$$

$$\frac{\alpha - b}{-f(b)} \boxed{\left(\lim_{x_k \to \alpha} \frac{f(x_k) - f(\alpha)}{x_k - \alpha} \right)}$$

$$\Rightarrow \lim_{k \to \infty} \frac{x_{k+1} - \alpha}{x_k - \alpha} = 1 - \frac{(b-\alpha) f'(\alpha)}{f(b)}$$

$\Rightarrow$ linear conv. if

(Constants okay!)

In Secant: we just use x_k, x_{k+1} to do linear approx!