Nonlinear Equations

- Where they come up (lots of places)

- Why they are hard to solve

- Bisection to solve them

Linear Equations

$A\underline{x} = \underline{b}$ (linear algebra

Extremely flexible! Eg.

$A\underline{x} = \underline{b}$ minimizes quadrat

Where have we seen $A\underline{x} = \underline{b}$?

- Normal Eq. (Best Approx.)
- Spline interpolation
- FEM
- Backward Euler Eq.
- Newton's Method.

Class.

- Sig Proc. (Comp. Sensing)
- Control
- Regression

:)

Eq.

-λ-vals

-Quaternions

Non-linear Eq.

⊕ BAD NAME

This describes what it is not. Not what they are!

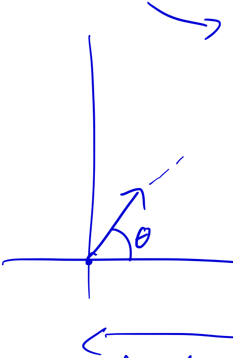
- We assume (at least) Continuous Eq.!
- Examples + Special Cases are key!

↳ synthesize Common Scenarios!

A Simple

Fixed, but w

Canon a



+ meters

Target.

Example!

e can move
ngle.

Launch at velocity
V_0

Where does projectile land?

↓ gravity!

Horizontal.
$d(\theta)$ = distance projectile goes

$$d(\theta) = \frac{2 v_0^2 \sin\theta \cos(\theta)}{g}$$

$g = 9.8\ m/s^2$

$V_0 = ?$

- "Military Canon"
  $V_0 = 1000\ m/s$ (2200 mph / 3000 kph)
- "potato Canon"
  140 m/s (300 mph / 500 kph)

Need to hit target

$$d(\theta) = t$$

$\Big\downarrow$ Canonicalize!

$$f(\theta) = 0$$

$$f(\theta) = d(\theta) - t$$

or

$$f(\theta) = t - d(\theta)$$

Terminology

Nonlinear Eq.

$$f(x) = 0$$

A solution is called a root or zero of f.

Why Canon Sys?

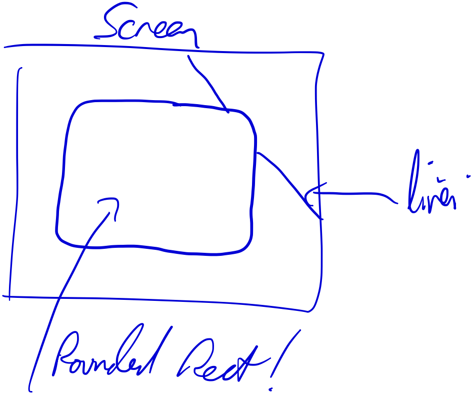
More

App

$\approx$

Examples

le loves Rounded Rectangle!



$= x^4 + y^4 \leq 1$

→ find intersection w/ line!

line:

$y = x + 1/2$

$$x^4 + (x + \tfrac{1}{2})^4 = 1$$

$\Rightarrow$

$$f(x) = x^4 + (x + \tfrac{1}{2})^4 - 1 = 0$$

Non linear Eq!

Loan Payments

Borrow 100K

Rate R

Applied in 12 increments $r = R/12$

Term 30 year => 360 months.

Monthly Payment p.

Month 0

100K

Month 1

$(100K)(1+r) - p$

Month 2

$((100K)(1+r) - p)(1+r)$

$- p$ =>

$(100K)(1+r)^2 - p(1+(1+r))$

$-$

360 months

$$(100k)(1+r)^{360} - p\boxed{(1+(1+r) + (1+r)^2 + (1+r)^3 + \dots + (1+r)^{359})}$$

↓

$$\frac{(1+r)^{360}-1}{r}$$

360 month =)

$$\boxed{(100k)(1+r)^{360} - p\frac{(1+r)^{360}-1}{r} = 0}$$

You have $500/month to use on a loan.

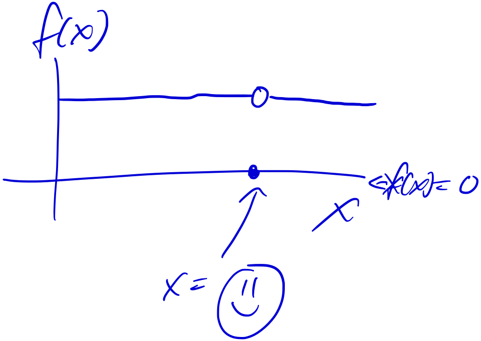
What is the largest rate you can tolerate?

↓

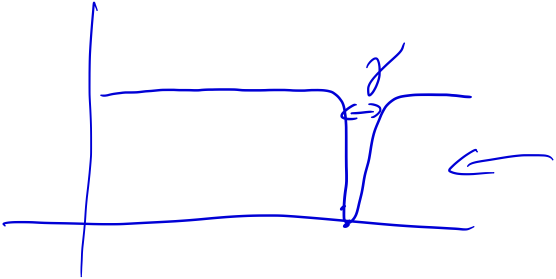
fix $p = 500$, set $= 0$, solve!

Why are Non-linear Eq. Hard?

(Why Continuity is critical, but not "Enough!")



Even Continuity Won't help



I can make δ as small as I want!

If I know I'm nearly a root, I can probably find it for a continuous function!

Next bit of Intro!

- System of Nonlinear Eq.

↙ multivar

↙ multivariate!

$$\underline{f}(\underline{x}) = 0$$

$$f : \mathbb{R}^n \to \mathbb{R}^m$$

↑ n variables

↑ m outputs!

* We will see lots in ODEs!

We already saw some!

$$\int_{-1}^{1} f(t)\,dt = w_0 f(t_0) + w_1 f(t_1)$$

Non-lin Eq. Setup:

Pick w_0, t_0, w_1, t_1 s.t.

we exactly integrate

$f(t) = 1$

$f(t) = t$

$f(t) = t^2$

$f(t) = t^3$

$$\int_{-1}^{1} 1 \, dt = 2 = w_0 + w_1$$

$$\int_{-1}^{1} t \, dt = 0 = t_0 w_0 + t_1 w_1$$

$$\int_{-1}^{1} t^2 \, dt = 2/3 = t_0^2 w_0 + t_1^2 w_1$$

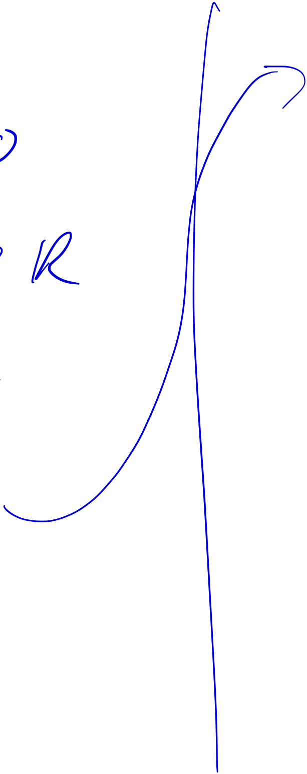
$$\int_{-1}^{1} t^3 \, dt = 0 = t_0^3 w_0 + t_1^3 w_1$$

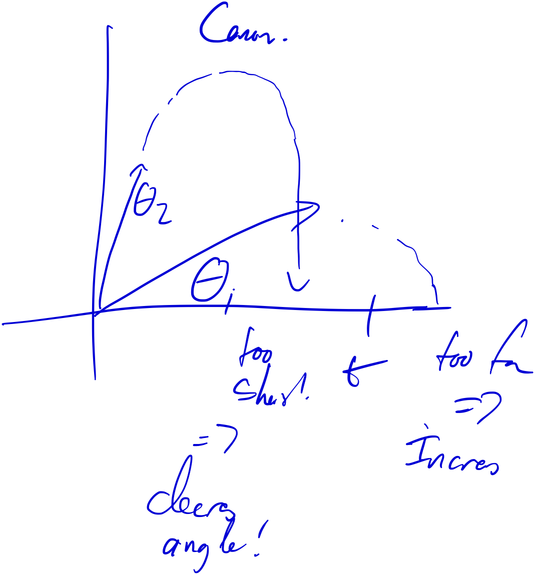
How to solve

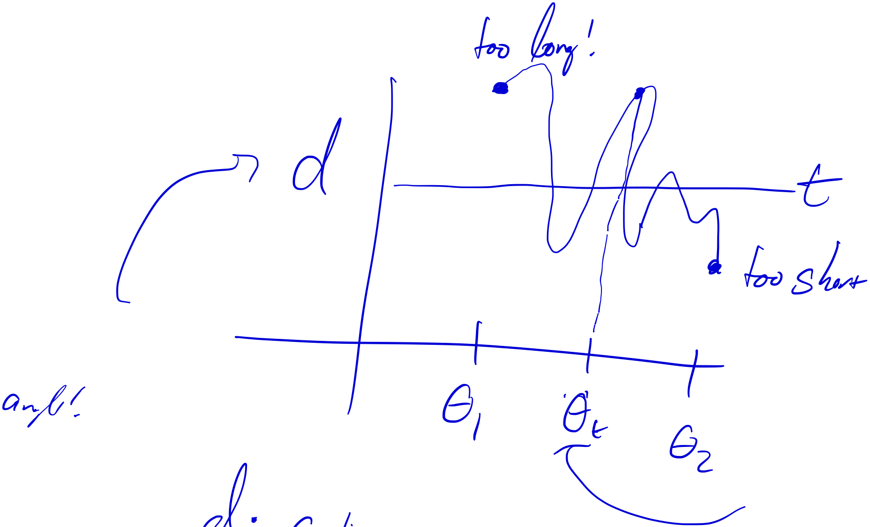
$f(x) = 0$

$f: \mathbb{R} \to \mathbb{R}$

- Guess, Check, Adapt!







d: Continuous $\Rightarrow$

$d(\theta) = 0$ for some $\theta \in [\theta_1, \theta_2]$

Bisection:

Test: $\theta = \frac{\theta_1 + \theta_2}{2}$

Then $d(\theta) > t \Rightarrow$ too far (still)

$\Rightarrow$ repeat bisect on

$[\theta_1, \theta_2)$

$\Rightarrow$ —

Alg: for $f(x)=0$.

Given: x_l, x_r

where

$f(x_l) f(x_r) < 0$.

"the sign of f at x_l, x_r differ!"

Test: $x_t = \frac{x_l + x_r}{2}$

If $f(x_t) > 0$, then replace either x_l, x_r s.t. $f(x_t) f(x_?) < 0$. →

At Each step, we reduce the Search Space by $\frac{1}{2}$

So we get Float Precision when

$$\left(\frac{1}{2}\right)^k (x_r - x_l) < \varepsilon(x_f)$$

$\Rightarrow$ ~~$= 52$~~ Steps to Solve

Max 64 Steps?

+ repeat!

$$x_f = \frac{x_\ell + x_r}{2}$$

is not the floating point midpoint!