Lecture 17

Interpolatory quadrature
Gaussian Quadrature
and why Orthogonal
polynomials come up
here!

And the method of
undetermined
coefficients

Quadrature => Numerical Integration

↳ Piecewise (composite)

↳ general quadrature   Much of class

$$\int_a^b f(t)\,dt = \sum f(t_i) w_i$$

↳ Another idea (un-determined Coeffcients or other stuff

$$\int_0^1 f(t)\,dt = \alpha_1 f(0) + \alpha_2 f(t)$$

pick α_1, α_2
to be exact for
some class of
functions f.

Quick Example of
undetermined coeffs

$$\int_0^1 f(x)\,dx = \alpha_1 f(0) + \alpha_2 f(1)$$

to be exact for linear function
f.

$\Rightarrow f(x) = ax + b$

$$\int_0^1 (ax+b)\,dx = \left(\frac{ax^2}{2} + bx\right)\Big|_0^1 =$$

$$\frac{1}{2}a + b$$

$\Rightarrow$

$$\frac{1}{2}a + b = \alpha_1 f(0) + \alpha_2 f(1)$$
$$= \alpha_1 b + \alpha_2 (a+b)$$

If $a = 0$, then $b = \alpha_1 b + \alpha_2 b \Rightarrow \alpha_1 + \alpha_2 = 1$

if $b = 0$ then

$\frac{1}{2}a = \alpha_2 a \Rightarrow$

$\alpha_2 = 1/2$

$\Rightarrow \alpha_1 = 1/2$

$$\underbrace{\frac{1}{2}f(0) + \frac{1}{2}f(1)}_{= \text{ trapezoid rule!}}$$

Example 2

$$\boxed{\int_0^1 \sqrt{x}\, f(x)\,dx = \alpha_1 f(0) + \alpha_2 \int_0^1 f(x)\,dx}$$

to be exact for linear f.

If exact for linear f, then
$f(x) = 1 \Rightarrow$

$$\int_0^1 \sqrt{x}\,dx = \frac{2}{3}x^{3/2}\Big|_0^1 = \frac{2}{3} =$$

For $f(x) = 1$

$$\int_0^1 x^{(0.5)}\,dx = \frac{2}{3}x^{3/2}\Big|_0^1 = \frac{2}{3}$$

$$= \alpha_1 \overset{=1}{f(0)} + \alpha_2 \underbrace{\int_0^1 1\,dx}_{=1}$$

$$= \alpha_1 + \alpha_2 = 2/3$$

For $f(x) = x$

$$\int_0^1 x^{3/2} = \frac{2}{5}x^{5/2}\Big|_0^1 = \frac{2}{5}$$

$$= \alpha_1 \overset{=0}{f(0)} +$$

$$\alpha_2 \int_0^1 x\,dx = \frac{\alpha_2}{2}$$

$$\Rightarrow \frac{2}{5} = \frac{\alpha_2}{2}$$

$$\alpha_2 = 4/5 \Rightarrow \alpha_1 = -$$

Terminology.
A quadrature rule is *interpolatory* when it integrates polynomials of degree $d = n - 1$ exactly.

A quadrature rule has *degree of exactness* d if it integrates polynomials of degree $\le d$ exactly.

Interpolatory is equivalent to exactness $\geq n - 1$.

Theorem. If we nodes and weights t_i, w_i which have degree of exactness $n-1$ (i.e. we are interpolatory) then

$$\boxed{\int_a^b \ell_j(t)\, dt = \sum_i \ell_j(t_i) w_i = w_j}$$

$$\int_a^b f(t)\,dt = \sum_{i=1}^n f(t_i) w_i$$

nodes weights

If $f(t)$ = poly of deg. $\le n-1$ $\iff$ Interpolation.

Pf If interpolatory:

$$\int_a^b \ell_j(t)\,dt = \sum_{i=1}^n \ell_j(t_i) w_i = w_j \ell_j(t_j) = w_j$$

$$\ell_j(t) = \begin{cases} 1 & t = t_j \\ 0 & \text{for other } t_i \ne t_j \end{cases}$$

Cause other terms to drop out!

$$\ell_j(t) = \frac{\prod_{k=1, k\ne j}^{n} (t - t_k)}{(t_j - t_k)}$$ ← a deg $n-1$ poly!

(We are given t_i's!)

Why important?

Let t_i, w_i be interpolation $\iff$

$$w_j = \int_a^b \ell_j(t)\,dt$$

So we have a weight expression!

Newton-Cotes:

Let t_i be uniformly spaced in $[a,b]$, then $w_j = \int_a^b \ell_j(t)\,dt$

$\Rightarrow$ Newton-Cotes Rule.

Newton Cotes

- fixes t_i
- picks w_i

$\Rightarrow$ deg of exactness $n-1$

Is deg. of exactness $> n-1$ possible? Yes!

Gaussian Quadrature

↳ pick nodes + weights together!

Intuition

$$\int_a^b f(t)\,dt = \sum_{i=1}^n w_i f(t_i)$$

w_i = n variable

t_i = n variables

$\Rightarrow$ $2n$ choices

- A few notes. Weighted integrals.

$$\boxed{\int_a^b f(t)\, w(t)\,dt =}$$

$$\int_a^b f(t)\,d\lambda(t)$$

- Let $w_n(t) = \prod_{k=1}^n (t - t_k)$ be the node polynomial.

Theorem: (Thm 3.2.1)

We have deg. of exactness $d = n+k-1$ iff

(a) the quadrature is interpolatory

(b) we have

$$\int_a^b w_n(t) p(t) w(t)\,dt = 0$$

for all polys $p(t)$ of deg $\le k-1$.

$f(t) = w_n(t) p(t)$

$\langle w_n(t), p(t) \rangle = 0$

orthogonal polys!

$$\langle f(t), g(t) \rangle = \int_a^b f(t) g(t) w(t)\,dt$$

Pf (3.2.1.)

deg. of exactness $n+k-1 \Rightarrow$ (a)(b)

deg. of exactness $n+k-1 \Rightarrow$

$$\int_a^b f(t) w(t)\,dt = \sum_{i=1}^n f(t_i) w_i$$

if $f(t)$ is a poly of deg $\le n+k-1$

Look at $f(t) = \boxed{p(t) w_n(t)}$ ← $p(t)$ deg $\le k-1$

$$w_n(t) = \prod_{i=1}^n (t - t_i)$$

$\Rightarrow$ deg n.

$p(t) w_n(t)$ is a poly of deg. $\le n+k-1$ $\Rightarrow$ Exact.

$$\sum_{i=1}^n p(t_i) w_n(t_i) w_i =$$

Is this 0?

$p(t_i)$ = ?? No?

$$w_n(t_i) = \prod_{k=1}^n (t_i - t_k) = 0$$

$\Rightarrow$ (b) holds!

(a, b) ⟹ Exactness.

Consider $f(t)$ = poly of deg $n+k-1$

I can write

$$f(t) = \underbrace{q(t)}_{\deg k-1}\overbrace{W_n(t)}^{\deg n} + \overbrace{r(t)}^{\deg n-1}$$

(remainder...)

(why? polynomial division.)

$\int_a^b q(t)\,W_n(t)\,w(t)\,dt = 0$ by (b)

$\int_a^b r(t)\,w(t)\,dt =$

$r(t) = f(t) - q(t)W_n(t)$

$r(t)$ is a poly of deg $n-1$ ⟹ Exact by interpolatory!

$$\int_a^b r(t)\,w(t)\,dt = \sum_{i=1}^{n} r(t_i)\,w_i = \sum_{i=1}^{n} f(t_i)\,w_i - \underbrace{\sum_{i=1}^{n} q(t_i)\,W_n(t_i)\,w_i}_{=0 \text{ by node poly } W_n(t_i)!}$$

$$\int_a^b f(t)\,dt = \underbrace{\int_a^b q(t)W_n(t)\,dt}_{=0} + \int_a^b r(t)\,dt = \sum f(t_i)\,w_i$$

⟹ Exactness = $n+k-1$

If $k = n$ called

Gaussian Quadrature (GQ)

- pick node polynomial $W_n(t)$ to be orthog to all polys of deg $\leq n$

Alg for GQ:

- find $W_n(t)$
- factor $W_n(t)$ to get $t_1, \ldots, t_n$
- Compute $w_1, \ldots, w_n$

Example

$w(t) = 1$

⟹ Legendre polys are orthogon w.r.t. $\int_{-1}^{1} f(t)\,g(t)\,dt = \langle f, g\rangle$

$\pi_0(t) = 1$

$\pi_1(t) = t$

$\pi_2(t) = t^2 - \frac{1}{3}$

$\boxed{w_j = \int_a^b \ell_j(t)\,w(t)\,dt.}$

nodes $= \pm\sqrt{1/3}$

weights $w_i = 1$.