Lecture 15

- Exam 1 - Still grading ☹
- Numerical derivatives

↳ polynomial interp..

↳ Central diff formula

  . Maybe algs?

- If time: Numerical integration

Goal: Estima

$f'(x_0)$

when given "

pointwise" Eval

Simple idea (

$f'(x) = \lim_{h \to 0}$

$\Rightarrow f'(x_0)$

te the derivative

oracle

oracle f

o of $f(x)$.

(last time)

$$\frac{f(x_0+h) - f(x_0)}{h}$$

$$\approx \frac{f(x_0+h) - f(x_0)}{h} \quad (h \text{ small})$$

h can't be too small though!

"Rule of thumb" ⇒

$$h \approx \sqrt{\varepsilon}$$

Gav

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Given

Goal

Think

Ex:

tschio approach

polynomial interp!

$f(x)$ at $x_0, x_1, \dots, x_n$

Approx $f'(x_0)$

w/ $x_0, x_1, \dots, x_n$ as

"nearby" x_0.

$x_1 = x_0 + h$

Why poly interp?

$f(x) = \underline{p_n(f; x}$

Interp. poly

$p_n(x)$

Lagrange $p_n(x)$

Newton → $p_n(x)$

$$r_n(x) = (x-x_0)(x-x_1)\cdots(x-x_n)\,\frac{f^{(n+1)}(\xi(x))}{(n+1)!}$$

$$\Big] + \underbrace{r_n(x)}$$

Residual / Error!

$$= \sum_{i=0}^{n} f_i \underbrace{\frac{\prod_{j \neq i}(x-x_j)}{\prod_{j \neq i}(x_i - x_j)}}_{\ell_i(x)}$$

$$) = f[x_0] + f[x_0,x_1](x-x_0) + f[x_0,x_1,x_2](x-x_0)(x-x_1) + \cdots$$

If f is smooth Enough, then this form is Exact!

$$f'(x) = p_n'(f, x) + r_n'(x)$$

| $p_n(x)$) Seems Easiest
to differentiate!

In fact, given setup,
we just want
$p_n'(x_0)$
(Newton)

$$p_n(x) = f[x_0] + f[x_0,x_1](x-x_0)$$
$$+ f[x_0,x_1,x_2]\underbrace{(x-x_0)(x-x_1)}$$
$$+ \cdots$$

$(fg)' = f'g + fg'$

product rule

$$p_n'(x_0) = 0 + f[x_0,x_1] + \underset{0}{(x-x_0)} \quad \underset{f[x_0,x_1,x_2](x_0-x_1)}{f[x_0,x_1,x_2](x-x_1)}$$

$$\left[\frac{\partial}{\partial x}\right.$$

$\Rightarrow$

$p_n'($

$$(x-x_0)(x-x_1)\cdots(x-x_n)\Big]'(x_0)$$

(by same process)

$$= (x_0-x_1)(x_0-x_2)\cdots(x_0-x_n)$$

$$x_0) = f[x_0,x_1] +$$

$$f[x_0,x_1,x_2](x_0-x_1) +$$

$$f[x_0,x_1,x_2,x_3](x_0-x_1)(x_0-x_2)$$

$$p_n'(x_0) = (x_0-x_1)(x_0-x_2)\cdots$$

$$+ \qquad + \qquad \cdots$$

product term

$$f'(x_0) = p_n'(f; x_0) +$$

$$(x_0-x_n)\,\frac{f^{n+1}(\xi)}{(n+1)!}$$

H

on $(x-x_0)$ vs $\underbrace{(x_0-x_1)\cdots}_{\text{Che}\ldots}$

e_n

If $H = \max_i |x_0 - x_i|$

then $e_n = O(H^n)$

Example:

$x_0, \quad x_0 + h = x_1$

$$p_1'(x_0) = f[x_0, x_1] = \frac{f(x+h}{h}$$

$$\left.\begin{matrix} x_0 & f_0 \\ x_1 & f_1 \end{matrix}\right\rangle \frac{f_1 - f_0}{x_1 - x_0} =$$

$$e_1 = (x_0 - x_1)\frac{f''(\xi)}{2}$$

$$\uparrow$$

$$= (x_0 - (x_0 + h))\frac{f''(\xi)}{2}$$

$$= -h\frac{f''(\xi)}{2}$$

$$f'(x_0) = p_1'(x_0) - h\frac{f''(\xi)}{2}$$

$$\frac{) - f(x)}{}$$

$$\frac{f(x+h) - f(x)}{h}$$

$$f'(x_0) - p_1'(x_0) =$$

$$=$$

If we know

f'' doesn

sign on

than because

we know the

Error!

$$-\frac{h f''(\xi)}{2}$$

$O(h)$

chang

$[x_0, x_0+h]$

$\xi(x_0) \in [x_0, x_0+h]$

Sign of the

Example 2

$x_0 = \text{given}$ $\qquad x_0 - h, \quad x_0 \quad x_0 + h$

$x_1 = x_0 - h \quad \rightarrow \quad x_{-1}, \; x_0, \; x_1$

$x_2 = x_0 + h$

$$\begin{array}{llll}
x_{-1} & f_{-1} & & \\
 & & \dfrac{f_0 - f_{-1}}{h} & \\
x_0 & f_0 & & \\
x_1 & f_1 & \dfrac{f_1 - f_0}{h} & \dfrac{\dfrac{f_1 - f_0}{h} -}{x_1 - x_{-1}} =
\end{array}$$

$f[x_{-1}, x_0]$

$$p_2'(x_0) = \cancel{f[x_0, x_1]} + \cancel{f[x_0, x_1, x_2]}(x_0 - x_{-1})$$

$= h$

$f[x_{-1}, x_0, x_1]$

$$= \frac{f_0 - f_{-1}}{h} + \frac{f_1 - 2f_0 + f_{-1}}{2h^{\cancel{2}}} \cancel{h}$$

$$p_2'(x_0) = \frac{f_0 - f_{-1}}{h}$$

$$= \cancel{\frac{f_0}{h}} - \frac{f_{-1}}{h} +$$

$$= \frac{f_1}{2h} - \frac{f_{-1}}{2h}$$

$$\frac{\frac{f_0 - f_{-1}}{h}}{2h}$$

$$= \frac{f_1 - 2f_0 + f_{-1}}{2h^2}$$

$$+ \frac{f_1 - 2f_0 + f_{-1}}{2h} =$$

$$\frac{f_1}{2h} - \cancel{\frac{2f_0}{2h}} + \frac{f_{-1}}{2h}$$

$$= \frac{f_1 - f_{-1}}{2h}$$

Called central diff formula

$$f'(x) \approx \frac{f(x+h) - f(x-h)}{2h}$$

$$e_h = O(h^2)$$

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Forward Diff:

$$f'(x_0) = \frac{f(x_0+h) - f(x_0)}{h} + O(h)$$

Backward Diff

$$f'(x_0) = \frac{f(x_0) - f(x_0-h)}{h} + O(h)$$

Central Diff.

$$f'(x_0) = \frac{f(x_0+h) - f(x_0-h)}{2h} + O(h^2)$$

Given:

$x_0, x_0+h, x_0+2h,$

$x_0-2h \Rightarrow$

plug + chug!

(Reasonable Algorithm)

Not Equally spread!

Gives a way to handle Errors in f!

Suppose $f_1^* = f_1 + \varepsilon_1$

$f_{-1}^* = f_{-1} + \varepsilon_{-1}$

Then $f'(x_0) = \dfrac{f_1^* - f_{-1}^*}{2h}$

$(x_1 = x_0 + h)$

$(x_{-1} = x_0 - h)$

$$-\frac{\varepsilon_1 - \varepsilon_{-1}}{2h} + \underbrace{e_2}_{= O(h^2)}$$

If

Overall Error

$$O\left(\frac{\varepsilon}{h}\right) + O(h^2)$$

To balance set

$$h = \varepsilon^{1/3}$$

Can you do better? Yes!

But much more "intricate"

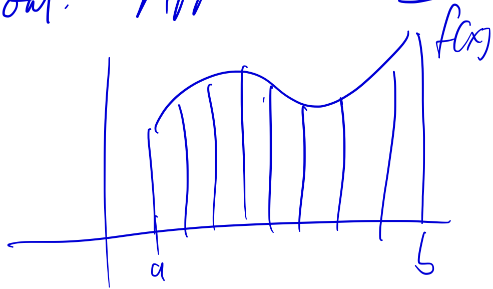
They use Cauchy's theorem.

$$f'(x_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-x_0)^2}\,dz$$

Numerical Integral

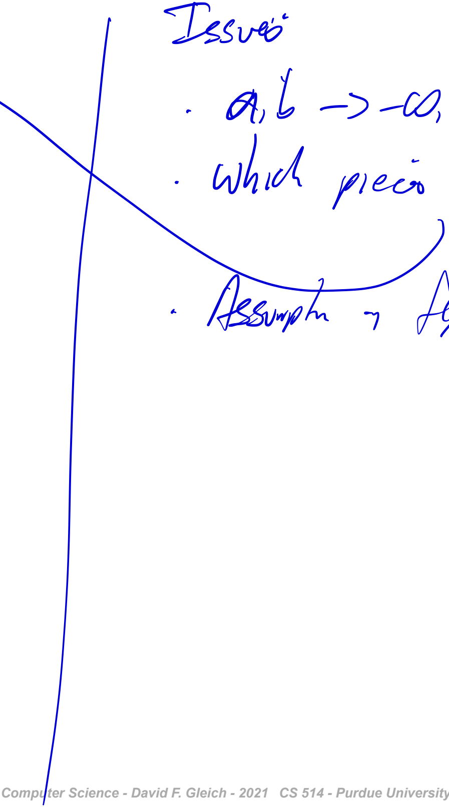
$$I = \int_a^b f(x)\,dx$$

Goal: Approximate I



Ideas

- Discrete Sums
- Using Derivative info
- Splines
- Fill w/ Pixels
- poly interp + analyze.
- Wolfram Alpha
- Anti-derivative



Issues:

- $a, b \to -\infty, \infty$
- which piece
- Assumptn → $f(x)$.