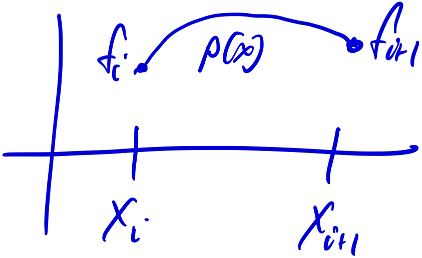
Splines

- piecewise approx on grid $x_0, x_1, \ldots, x_n$
- Overall approx is "smooth"
- Each piece is a poly.

Cubic Splines (Er



$$\begin{aligned} p(x_i) &= f_i \\ p(x_{i+1}) &= f_{i+1} \\ p'(x_i) &= m_i \\ p'(x_{i+1}) &= m_{i+1} \end{aligned} \Bigg]$$

piece!)

Q: what are M_i's?

- Idea 1.

  $M_i = f'(x_i)$

- Idea 2

  M_i for smoothness.

$$p''_{[i,i+1]}(x_{i+1}) = p''_{[i+1,i+2]}(x_{i+1})$$

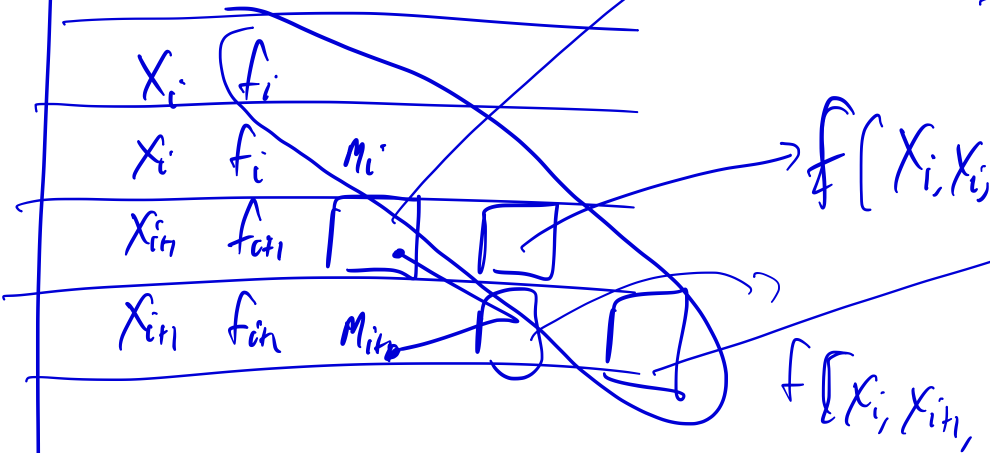
System of linear Eq.

||

2nd deriv

"Match" @ End points.

Let's do Hermite poly for general m_i.



$$\frac{f[x_{i+1}] - f[x_i]}{(x_{i+1} - x_i)}$$

$$\to f[x_i, x_i, x_{i+1}, x_{i+1}] = \frac{f[x_i, x_{i+1}, x_{i+1}] - f[x_i, x_i, x_{i+1}]}{(x_{i+1} - x_i)}$$

$$x_{i+1}] = \frac{f[x_i, x_{i+1}] - m_i}{(x_{i+1} - x_i)}$$

$$x_{i+1}] = \frac{m_{i+1} - f[x_i, x_{i+1}]}{(x_{i+1} - x_i)}$$

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$$p(x) = f_i + \underset{M_i}{\cancel{f[x_i, x_i]}}(x - x_i)$$

$$+ f[x_i, x_i, \cancel{x} x_{i+1}](x - x_i)^2$$

$$+ f[x_i, x_i, x_{i+1}, x_{i+1}](x - x_i)^2 (x - x_{i+1})$$

What is

$p''(x_i)$?

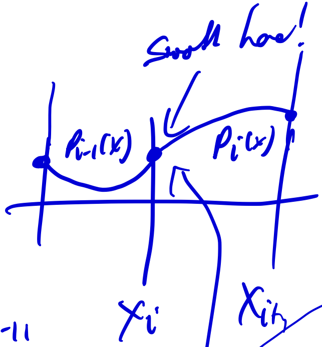
$p''(x_{i+1})$?

Suppose (How to Algeb

$$p(x) = a(x - x_i)^3 + b(x - x_i)^2 + c(x - x_i) + d$$

Recall

a, b, c, d are linear function of M_i, M_{i+1}.



get this? (Taylor!)

$$\boxed{\begin{aligned} p''(x_i) &= 2b \\ p''(x_{i+1}) &= 6a(x_{i+1}-x_i) + 2b \end{aligned}}$$

⇒ linear in M_i's!

Let

a_i, b_i be the

Coeffs for

x_i, x_{i+1} piece!

$p_i(x)$ be the x_i, x_{i+1} piece.

We want

$$\boxed{\overset{2b_i}{\overset{\parallel}{p_i''(x_i)}} = \overset{6a_{i-1}(x_i - x_{i-1}) + 2b_{i-1}}{\overset{\parallel}{p_{i-1}''(x_i)}}}$$

to get 2nd order smooth.

But this is using a_i, b_i!

Not M_i!

But we have

$$a_i = s_i^{(1)} M_i + s_i^{(2)}$$
$$+ s_i^{(3)}$$

$$b_i = f_i^{(1)} M_i + f_i^{(2)}$$
$$+ f_i^{(3)}$$

clo

$)M_{i+1}$

M_{i+1}

$$2b_i = 6a_{i-1}(x_i - x_{i-1}) + 2b_{i-1}$$

$\Downarrow$ Cancel 2

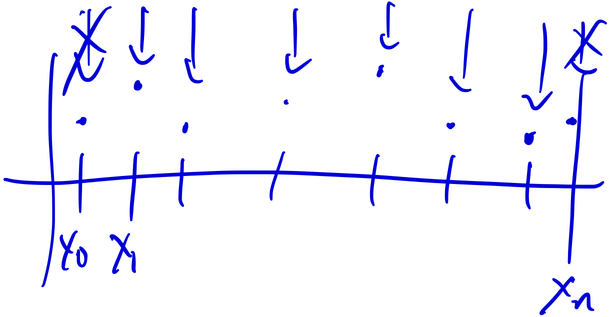
$$t_i^{(1)} M_i + t_i^{(2)} M_{i+1} + t_i^{(3)} =$$

$$\left(3 s_{i-1}^{(1)} M_{i-1} + 3 s_{i-1}^{(2)} M_i + 3 s_{i-1}^{(3)}\right)(x_i - x_{i-1}) + t_{i-1}^{(1)} M_{i-1} + t_{i-1}^{(2)} M_i + t_{i-1}^{(3)}$$

$$\boxed{r^{(1)} M_{i+1} + r^{(2)} M_i + r^{(3)} M_{i-1} = r^{(4)}}$$

Nice linear Eq!

sed form
soln for
$r^{(1)}$!



Unkn M_i's $\Rightarrow n+1$

But we only have

$n-1$ Eq.!

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2 more
deg of
freedom!

Pick M_0, M_n based
on $f'(x_0), f'(x_n)$

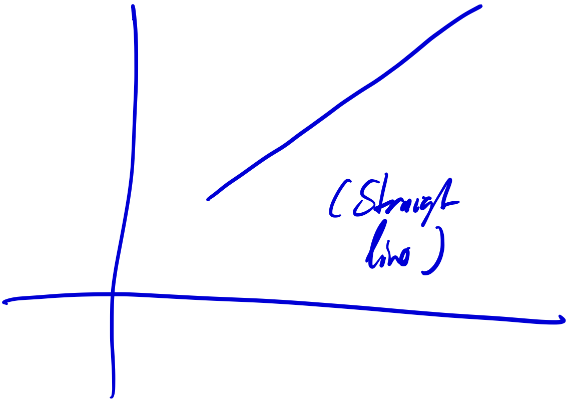
Pick ~~$M_0, M_n = 0$.~~
$f''(x_0), f''(x_n) = 0$.

Make $S(x)$ 2nd deriv.
Cont. @ x_1, x_{n-1}

Chapter 3

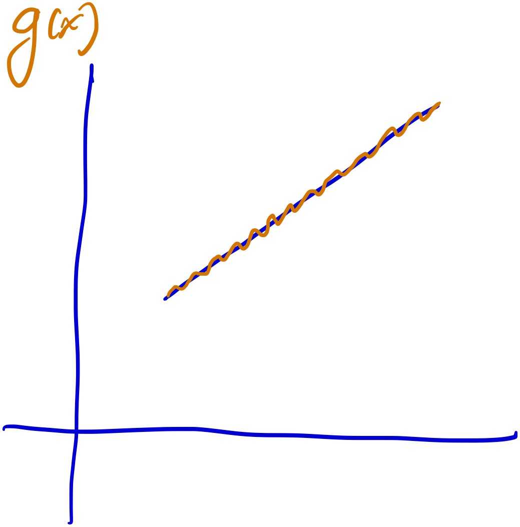
- Derivatives!
- Derivatives are "fundamentally" ill conditioned!

$f(x) =$



f, g are

f', g' are



close !

not (always) close!

Maybe we should Expect Difficulties

How to get

$f'(x)$?

- Automatic Differen

"Simple Enough"

↳ given program a compiler can

for you!

!

tion (No approx!)

for $f(x)$

do this

- $$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Idea: Approx.

$$f'(x) \approx \frac{f(x+h) - f(x)}{h}.$$ → How good?

(Just Eval function twice!)

Easy! w/ Oracle access.

We ca study!

By Taylor Series ...

$\xi(x) \in [x, x+h]$

$$f(x+h) = f(x) + h f'(x) + \frac{h^2}{2} f''(\xi(x))$$

$$\Downarrow$$

$$f'(x) = \frac{f(x+h) - f(x)}{h} + \underbrace{\frac{h}{2} f''(\xi(x))}_{O(h)}$$

If $(f''(\xi)$

$h =$

Should b

$1 \leq 10000$,

10^{-15}

good!

But it isn't!

Let's assume (for simplicity) we just make errors in $f(x)$, $f(x+h)$

Compute

$$\frac{f(x+h)(1+\varepsilon_1) - f(x)(1+\varepsilon_2)}{h} =$$

$$f'(x) + O(h)$$

$$\|$$

$$|\text{true} - \text{computed}| = O\left(\frac{\varepsilon}{h}\right)$$

$$\|$$

$$\boxed{\text{true} + O\left(\frac{\varepsilon}{h}\right)}$$

$$\|$$

$$\frac{f(x+h) - f(x)}{h} + \frac{\varepsilon_1 (f(x+h)) - \varepsilon_2 f(x))}{h}$$

$|$

$|f'(x$

$(\text{Erro}$

as

$f'(x) + O(h) -$

$\text{Comput}) | =$

$O\left(\frac{\varepsilon}{h}\right)$

$\min_{h} \iff h =$

$\iff h =$

(More refined an

possible!)

$o) \doteq \text{Comput} | = O\left(h + \frac{\varepsilon}{h}\right)$

$r)$

$h \to 0, \quad O\left(h + \frac{\varepsilon}{h}\right) \to \infty \,!$

$$\frac{\varepsilon}{h}$$

$$\sqrt{\varepsilon}$$

[illegible]