# Lecture 12

- Quick Hermite Example
- Piecewise Interpolation
  - ↳ linear
  - ↳ best approx
  - ↳ splines

⊛ Exam Comin up!

## Quick Ex of Hermite

find $p$ s.t.

$p(x_0) = f_0$

$p(x_1) = f_1$, $p'(x_1) = f_1'$

$p(x_2) = f_2$

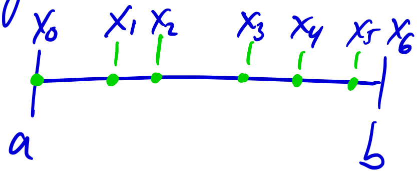
$$\begin{tikzcd}
x & f \\
x_0 & f_0 = c_0 \arrow[dr, dash] \\
 & & f[x_0,x_1] = c_1 \arrow[dr, dash] \\
x_1 & f_1 \arrow[ur, dash] \arrow[dr, dash] & & f[x_0,x_1,x_1] = c_2 \arrow[dr, dash] \\
 & & f' \arrow[ur, dash] \arrow[dr, dash] & & f[x_0,x_1,x_1,x_2] = c_3 \\
x_1 & f_1 \arrow[ur, dash] \arrow[dr, dash] & & f[x_1,x_1,x_2] \arrow[ur, dash] \\
 & & f[x_1,x_2] \arrow[ur, dash] \\
x_2 & f_2 \arrow[ur, dash]
\end{tikzcd}$$

$$f[x_0,\ldots,x_k] = \frac{f[\overbrace{x_1,\ldots,x_k}^{\text{r "dupes allowed"}}] - f[x_0,\ldots,x_{k-1}]}{(x_k - x_0)}$$

$$p(x) = f_0 + c_1(x - x_0) + c_2(x - x_0)(x - x_1)$$
$$+\ c_3(x - x_0)\underbrace{(x - x_1)(x - x_1)} = (x-x_1)^2$$

# Piecewise Interpolation / Approx.

Region:

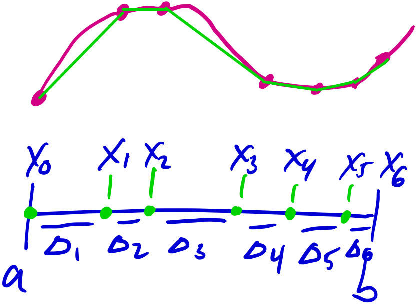


Called a subdivision of $[a,b]$
or a grid on $[a,b]$

- Don't have to be Equally Spaced! (But often are!)

$\{x_i\}$ is a subdivision of

Example: Piecewise linear



Let $\Delta_i = x_i - x_{i-1}$

$[a,b]$

Then if $\max_i |\Delta_i|$ is small, we "expect" piecewise linear approx. to be good.

↳ Intuition from computer plotting

What is piecewise linear interpolant?

Given $\{x_i\}$ and $\{f_i\}$

$$\boxed{S(x) = f_i + f[x_i, x_{i+1}](x - x_i) \quad \text{if } x_i \le x \le x_{i+1}}$$

Error

$$f(x) - S(x) \qquad = (x - x_i)($$

$$x \in [x_i, x_{i+1}]$$

$$|f(x) - S(x)| \le |(x - x_i)(x -$$

"$f[x_i, x_{i+1}, x]$"

$$\|$$

$$(x - x_{i+1}) \frac{f''(\xi(x))}{2}$$

$$x_{i+1}) \mid \max_{\substack{x \in \\ [x_i, x_{i+1}]}} \frac{|f''(x)|}{2}$$

Claim: (won't prove)

$$|(x-x_i)(x-x_{i+1})| \le \frac{(x_{i+1}-x_i)^2}{4}$$

$$= \Delta_i$$

Why nice?

$$\underset{x \in [x_i, x_{i+1}]}{|f(x) - s(x)|} \le \frac{|\Delta_i|^2}{4} \max_{x \in [x_i, x_{i+1}]} \frac{|f''(x)|}{2}$$

$$\|f(x) - s(x)\|_\infty = \frac{\|\Delta\|_\infty^2}{8} \cdot \|f''\|_\infty$$

$$\|\Delta\|_\infty = \max_i |\Delta_i|$$

$$\Delta = \begin{bmatrix} \Delta_1 \\ \vdots \\ \Delta_n \end{bmatrix}$$

Error $\to 0$

$\Delta_i \to 0$

(Exactly our intuition!)

## More generally

(Notation)

$$S_M^K(\Delta)$$

Set of functions on a grid $\Delta$ where each piece is a poly of degree $M$ and the function is $K$ times continuous diff.
("Smoothness $K$")

Called

Spline functions

$\{ S : \boxed{S \in C^K[a,b]}$

piece! $S|_{x_i, x_{i+1}} \in$ poly of

the overall function s!

deg m }

If $K = m$, then

$$S_m^m = \text{polys of degree } m.$$

(no piece-wise behavior.)

If $K = -1$, we allow discontinuities!

What is piecewise linear?

$$S_1^0 - \text{piecewise linear.}$$

## Fact 1

- piecewise interpolation nodes of gives alm approximati

$S_1^0$.

linear

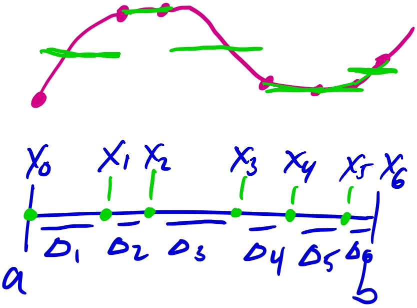
at the

$x_i$

...st best

...i ...

$S_0^{-1}$ = piecewise Constant.



Let $S_1$

piecewise

on $\Delta$

Let S

∞-norm.
approx. f

Then $\| f$

(piecewise

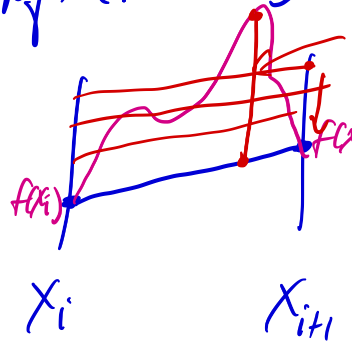
[f] be the interpolant to f

be the best to f in $S_1^0$.

$-S_1[f]\|_\infty \le 2\|f-s\|_\infty$

linear is w/in factor 2 of best!)

Pf: (Picture!)



$\max |S_1(x) - f(x)|$

sin)

↓

Improve by moving line!

↓

But at some point, the other approx. get worse => middle => factor 2.

# Pf (formal)

$f - S_1[f] =$

$f - S_1[f] + S_1[S] - S_1$

↑

piecewise linear approx. to $S$

$S_1[S]$ = piecewise linear interp. $S$.

$) = f - s \underbrace{+ S_1[s] - S_1[f]}_{= S_1[s-f]}$

$[s)$

$= f - s + S_1[s-f]$

Piecewise linear on Δ to $s$

−

piecewise linear on Δ to $f$

ie.

piecewise linear to $s-f$ on Δ.

$- \,/ = s.$

$\|f - S_1[f]\|_\infty$

$\le \|f - s\|_\infty + \boxed{\|S_1[s-f]\|_\infty}$

Is $\|S_1[s-f]\|_\infty$ ← piecewise

$\|s - f\|_\infty$ ?

yes!

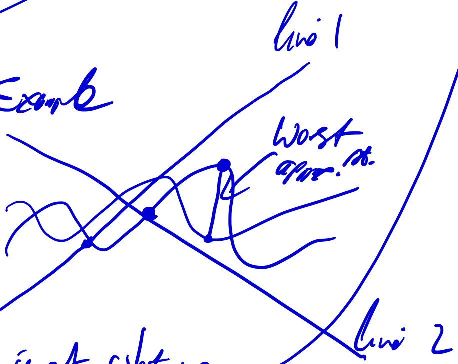
$\| f - S_j[f] \| \le$

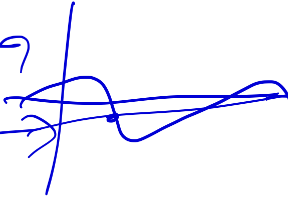
$2 \cdot \| f - S \|$

approx.

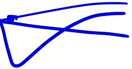
$\le$

Example





This is not what we want!



$S_3^1(\Delta)$

$S_3^2(\Delta)$

Cubic Spline!

Idea!

$M_i = f$

(more ideas,

I don't

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given $x_i$, $f_i$ and

values $m_i$

s.t.

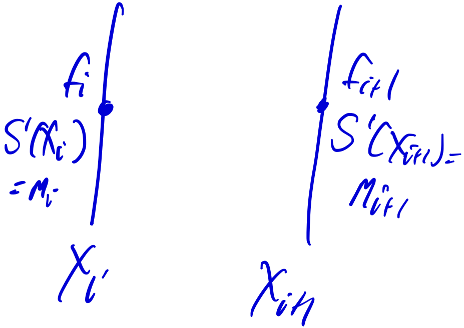
$$\boxed{S'(x_i) = m_i}$$

:

$'(x_i)$

hey!
have $f'(x_i)$

Goal: Preserve
Cubic ✓



What is the cubic
$S(x)$ on
$[x_i, x_{i+1}]$?

Hermite Interpolant!

$x_i \quad f_i$

$x_i \quad f_i \quad m_i$

$x_{i+1} \quad f_{i+1} \quad f[x_i; x_{i+1}] \quad f[x_i; x_i; x_{i+1}]$

$x_{i+1} \quad f_{i+1} \quad m_i \quad f[x_i; x_{i+1}, x_{i+1}]$

$f[x_i, x_i, x_{i+1}, x_{i+1}]$