# Lecture 11

- Polynomial Interp Review
- Newton
- Hermite

Given

(distinct!) $x_0, \ldots, x$

$f_0, \ldots, f_n$

$$p_n(x) = \sum_{i=0}^{n} f_i \ell_i($$

If $f(x_i) = f_i$

and $f$ is smooth

$$f(t) - p_n(t)$$

— points

n — nodes

— function values.

o) $$\ell_i(x) = \prod_{\substack{j \ne i \\ j=0}}^{n} \frac{(x - x_j)}{(X_j - X_i)} = \begin{cases} 1 & @ \; x_i \\ 0 & @ \; x_j \ne x_i \end{cases}$$

Enough

Taylor! 0 at $x_0, \ldots, x_n$

$$= \boxed{\frac{1}{(n+1)!} f^{(n+1)}(\xi(x))} \; \boxed{\prod_{i=0}^{n} (x - x_i)}$$

$x \in [a,b]$

$\xi \in [a,b]$

Newton

$p_n($

# Forms

$$x) = a_0 + a_1(x-x_0) +$$
$$a_2(x-x_0)(x-x_1) +$$
$$a_3(x-x_0)(x-x_1)(x-x_2) +$$
$$a_4(x-x_0)(x-x_1)(x-x_2)(x-x_3) +$$
$$\cdots$$
$$a_n \prod_{i=0}^{n-1}(x-x_i).$$

$$p_n(x_0) = a_0 + 0$$
$$+ 0$$
$$+ 0 + 0 -$$
$$+ 0 .$$

$$a_0 = p_n(x_0) = f_0$$

$$p_n(x_1) = a_0 + a_1(x_1 - x_0) = f_1$$

$$a_1 = \frac{f_1 - a_0}{x_1 - x_0}$$

$$\vdots$$

$a_2$ – $a_n$ similar!

We like a different notation on this!

Book

$[x_0, \ldots, x_n]f =$

Us

$f[x_0, \ldots, x_n]$

$a_i = f[x_0, \ldots, x_i]$

$$f[x_0, \ldots, x_i] = \frac{f[x_1, \ldots, x_i] - f[x_0,}{(x_i - x_0)}$$

Base Case

$f[x_i] = f_i$

$a_0 = f[x_0] = f_0$

$a_1 = f[x_0, x_1] = (f[x_1]$

$$\ldots, x_{i-1}]$$

$$\ldots - f(x_0))/(x_1 - x_0)$$

$$\frac{f_1 - f_0}{(x_1 - x_0)} = a_1$$

vs.

$$a_1 = \frac{f_1 - a_0}{x_1 - x_0}$$

$= ??$

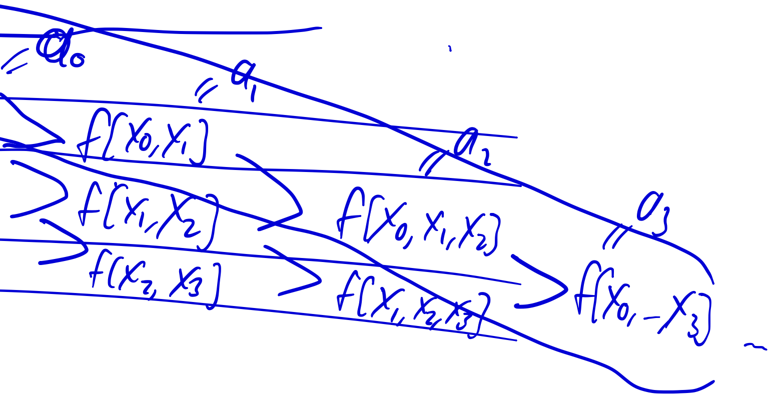
Yes! Equal
Because $f_0 = a_1$

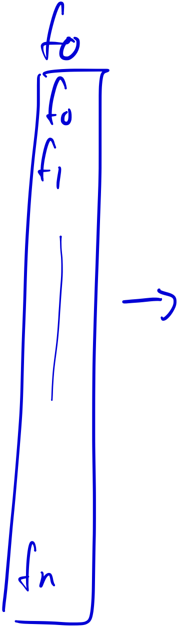
Why neat?
There is a ta

| $x$ | $f$ |
|---|---|
| $x_0$ | $f_0 = f[x_0]$ |
| $x_1$ | $f_1 = f[x_1]$ |
| $x_2$ | $f_2 = f[x_2]$ |
| $x_3$ | $f_3 = f[x_3]$ |
| $x_4$ | $\vdots$ |
| $x_5$ | |
| $x_6$ | |

Q: Do we
A: No, but

ble!





need $O(n^2)$ memory?

we will not discuss that optimization.

$$\begin{bmatrix} \boxed{f_0} \\ f[x_1,x_2] \\ f[x_2,x_3] \\ \vdots \\ \vdots \end{bmatrix} \rightarrow \begin{bmatrix} a_0 \\ a_1 \\ f[x_0,x_1,x_2] \\ \\ \\ \end{bmatrix}$$

# Hermite Interpolation

Given

$x_0 \quad x_1 \quad - \quad x_n$

Also

$f_0 \quad f_1 \quad - \quad f_n$

Also some derivatives

$M_0 \quad M_1 \quad - \quad M_n$

= # of derivatives @ Each point!

+1

i.e.:

|  |  |  |  |  |  |
|---|---|---|---|---|---|
| $x_0$ | $x_1$ | $x$ | $x_3$ | $x_4$ | $x_5$ |
| $f_0$ | $f_1$ | $f_2$ | $f_3$ | $f_4$ | $f_5$ |
| $f_0'$ |  |  | $f_3'$ | $f_4'$ | $f_5'$ |
|  |  |  | $f_3''$ |  | $f_5''$ |
|  |  |  |  |  | $f_5'''$ |

~~$M_0=1$ $M_1=0$ $M_2=0$ $M_2=2$ $M_4=1$ $M_5=3$~~

$M_0=2$ $M_1=1$ $M_2=1$ $M_3=3$ $M_4=2$ $M_5=4$

Q: what degree p
interpolates?
(w/ the same
derivatives?

|  |  |  |  |
|---|---|---|---|
| $x_0$ | $x_1$ | $x_2$ | $x$ |
| $f_0$ | $f_1$ | $f_2$ | $f_3$ |

$\longrightarrow$ 5+1 degree

$\Rightarrow$ A: $\left(\sum_{i=0}^{n} m_i\right) - 1$

oly

3 $x_4$ $x_5$

$f_4$ $f_5$

interpolant.

= deg 4

Interpolant!

Alg for Hermite
Duplicate points
in Newton Table.

| $x$ | $f$ | | |
|---|---|---|---|
| $x_0$ | $f_0$ | | |
| $x_0$ | $f_0$ | $f'(x_0)$ | |
| $x_1$ | $f_1$ | • | $f[x_0, x_0, x_1]$ |
| $x_2$ | $f_2$ | • | $f[x_0, x_1, x_2]$ |
| $x_3$ | $f_3$ | • | $f[x_1, x_2, x_3]$ |
| $x_3$ | $f_3$ | $f'(x_3)$ | $f[x_2, x_3, x_3]$ |
| $x_3$ | $f_3$ | $f'(x_3)$ | $f''(x_3)/2$ |
| $x_4$ | $f_4$ | • | $f[x_3, x_3, x_4]$ |
| $x_4$ | $f_4$ | $f'(x_4)$ | $f[x_3, x_4, x_4]$ |

Q: Is this correct?

A: Yes! Lets prove it!

(corrected!)

$$f[\underbrace{x_0,\ldots,x_0}_{n+1 \text{ times}}] = \frac{f^{(n)}}{n!}(x_0)$$

$$= \cancel{\frac{f^{(n)}}{(n+1)!}(x_0)}$$

# Simple Example

Given $x_0$, $M_\partial = 4$

$f_0, f_0', f_0'', f_0'''$

We want

$$p(x) = ax^3 + bx^2 + cx + d$$

$$p'(x) = 3ax^2 + 2bx + c$$
$$p''(x) = 6ax + 2b$$
$$p'''(x) = 6a$$

s.t. $p(x_0) = f_0 =$

$p'(x_0) = f_0'$ $\Rightarrow$ -

$p''(x_0) = f_0'' = 6ax_0 + 2b \Rightarrow b = \dfrac{f_0'' - 6ax_0}{2}$

$p'''(x_0) = f_0''' = 6a$ $\quad (p'''(x) = 6a)$

$\Rightarrow a = f_0'''/6$

Goal: limit argumen

$$f[\underbrace{x_0, \ldots, x_0}_{n+1 \text{ time.}}]$$

Note: Newton Polys are Recursive

$$\underbrace{p_n(x)}_{n^{th}} = \underbrace{p_{n-1}(x)}_{n-1^{th}} + \underbrace{a_n \prod_{i=0}^{n-1}(x-x_i)}_{n \text{ deg poly.}}$$

Let's think abou

adding a point

$t$:

$$p_{n+1}(x) = p_n(x) +$$

$$p_{n+1}(t) = f(t) = p_n$$

$$\boxed{f(t) - p_n(t)} =$$

$t$

$\rightarrow$ From start.

$$\overset{\overset{a_{n+1}}{\|}}{\overbrace{f[x_0, \dots, x_n, t]}} \prod_{i=0}^{n} (x - x_i)$$

$$( \quad (x - t) \quad ) \quad !$$

$$(t) + f[x_0, \dots, x_n, t] \prod_{i=0}^{n} (t - x_i)$$

$$f[x_0, \dots, x_n, t] \prod_{i=0}^{n} (t - x_i)$$

$$f(t) - p_n(t)$$

$$f(t) - p_n(t) = f[x_0, \dots,$$

Suppose $(x_0,$

but

Taylor !

0 at $x_0, \dots, x_n$

$$= \boxed{\frac{1}{(n+1)!} f^{(n+1)}(\xi(t))} \underset{=}{\boxed{\prod_{i=0}^{n}(t-x_i)}} \quad \boxed{\begin{array}{l} t \in [a,b] \\ \xi \in [a,b] \end{array}}$$

$$x_n, t] \prod_{i=0}^{n}(t-x_i)$$

$$\frac{1}{(n+1)!} f^{(n+1)}(\xi(t)) \prod_{i=0}^{n}$$

$$\overset{t \in [a,b]}{\dots, x_n, t)} \longrightarrow x_0$$

ll stay distinct.

$$f[x_0, \dots, x_n, t] \cancel{\prod_{i=0}^{n}(t}$$

Goal

(Same!)

$\lim x_i, t \to x_0$

$(t - x_i)$

$= \frac{1}{(n+1)!} f^{(n+1)}(\xi(t)) =$

$\xi(t) \to x_0$

$- x_i)$

$f[x_0, \ldots, x_n, t]$

$\Rightarrow \frac{1}{(n+1)!} f^{(n+1)}($

~~$f[\underbrace{x_0, \ldots, x_0}_{n+1 \text{ time}}] = \frac{f^{(n)}}{(n+1)!}(x_0)$~~

Wrong Goal

$$f[x_0, \ldots, x_0]$$

$$\frac{f^{(n)}}{n!}(x_0)$$

$$x_0) = f[\underbrace{x_0, \ldots x_0}_{n+1}, x_0)$$

, (