HERMITE INTERPOLATION VIA NEWTON TABLES

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These notes are based on Chapter 2 in Gautschi's Numerical Analysis textbook.

POLYNOMIAL INTERPOLATION FACTS

Given a distinct set of points or nodes $x_0, \ldots, x_n$, and a value $f_0, \ldots, f_n$ at each point, there is a unique polynomial of degree n that interpolates the values f at those points.

The Lagrange form of the polynomial is

$$p_n(x) = \sum_{i=0}^{n} f_i \ell_i(x) \qquad \ell_i(x) = \frac{\prod_{j \neq i}(x - x_j)}{\prod_{j \neq i}(x_i - x_j)}$$

The error in this expression (for a sufficiently smooth function f) is

$$p_n(x) - f(x) = \frac{1}{(n+1)!} f^{(n+1)}(\xi) \prod_{j=0}^{n} (x - x_j).$$

Alternative ways to write the polynomial include: the Barycentric form and the Newton interpolant.

The Newton form is nice when we need to consider using *derivative information* in the interpolation.

The idea motivating Hermite interpolation is that $f[x_0, \ldots, x_k] = \frac{1}{k!} f^{(k+1)}(x_0)$ in the limit as all the points go together.

ADDING A POINT TO NEWTON

Consider what happens when adding a point to a Newton interpolation.

Here, we use the recursive form of the Newton Polynomial

$$p_n(x) = p_{n-1}(x) + a_n(x - x_0) \cdots (x - x_n - 1)$$

Then if we add a new point t we get

$$p_{n+1}(x) = p_n(x) + f[x_0, \ldots, x_n] \prod_{i=0}^{n} (x - x_i).$$

We then know that

$$p_{n+1}(t) = f(t) = p_n(t) + f[x_0, \ldots, x_n] \prod_{i=0}^{n} (t - x_i).$$

Then we can write

$$f(t) - p_n(t) = f[x_0, \ldots, x_n] \prod_{i=0}^{n} (t - x_i).$$

This is the error expression at t. (Except, this depends on $f(t)$ so this isn't saying much.)

But for a sufficiently smooth f we have

$$f(t) - p_n(t) = f[x_0, \ldots, x_n] \prod_{i=0}^{n} (t - x_i) = \frac{1}{(n+1)!} f^{(n+1)}(\xi) \prod_{j=0}^{n} (x - x_j)$$

for some ξ in the region containing the x's.

MOVING THE POINTS TOGETHER

The idea motivating Hermite interpolation

A QUICK EXAMPLE

The key fact is that we wish to replace

$$f[\underbrace{x_i, \ldots, x_i}_{n+1\text{times}}] = \frac{1}{n!} f^{(n+1)}(x_i).$$

so

$$f[x_i, x_i] = f^{(1)}(x_i) = f'(x_i)$$

and

$$f[x_i, x_i, x_i] = \frac{1}{2} f^{(2)}(x_i) = f''(x_i)$$

(a good to remember the coefficients is just Taylor series!)

- $x_0, x_1, x_2, x_3, x_4, x_5$
- f_0, f_0'
- f_1
- f_2
- f_3, f_3', f_3''
- f_4, f_4'
- f_5, f_5, f_5'', f_5'''

Then we setup a table

x	f	
x_0	f_0	
x_0	f_0	f_0'
x_1	f_1	X
x_2	f_2	X
x_3	f_3	
x_3	f_3	
x_3	f_3	
x_4	f_4	
x_4	f_4	
x_5	f_5	
x_5	f_5	
x_5	f_5	
x_5	f_5	