ERROR ANALYSIS FOR NUMERICAL COMPUTER METHODS

David F. Gleich

February 2, 2021

These notes are based on Chapter 1 in Gautschi's Numerical Analysis textbook.

THE SETUP

You have a problem P that you wish to solve

$$\underbrace{\begin{array}{ccccc} \mathbf{x} & & & & \mathbf{y} \\ \text{Input } \textit{or} & \rightarrow & \boxed{\mathbf{f}} & \rightarrow & \text{Output } \textit{or} \\ \text{Data} & & & & \text{Result} \end{array}}_{\text{the problem } P}$$

The goal in error analysis to is understand how well what you do on the computer approximates what you set out to do.

THE DIAGRAM WE ARE WORKING TO UNDERSTAND

$$\begin{tikzcd}
\mathbf{x} \in \mathbb{R}^m \arrow[r, "\mathbf{f}"] \arrow[d, "\text{Put } \mathbf{x} \text{ on machine}"'] & \mathbf{y} \in \mathbb{R}^n \\
\mathbf{x}^* \in \mathbb{M}^m \arrow[r, "\mathbf{f}"'] \arrow[dr, "\mathbf{f}_A"] & \mathbf{y}^* \in \mathbb{R}^n \\
 & \mathbf{y}_A^* \in \mathbb{M}^n \\
\mathbf{x}_A^* \in \mathbb{R}^n \arrow[ur, "\mathbf{f}"] &
\end{tikzcd}$$

ERROR ANALYSIS IS CONNECTING THE DOTS.

We need to understand how $\mathbf{y}_A^*$ relates to $\mathbf{y}^*$ and how $\mathbf{y}^*$ and $\mathbf{y}$.

1. $\mathbf{y}_A^*$ vs $\mathbf{y}^*$ – this is an algorithm question that turns into a math question (this class!
2. $\mathbf{y}^*$ vs $\mathbf{y}$ – this is a math question – just calculus! (last class!); we showed that *condition numbers* give us the sensitivity of the function $\mathbf{y}$ to any type of perturbation.

THE ERROR BOUND

Consider the actual error with a Taylor bound

$$\mathbf{f}(\mathbf{x}_A^*) - \mathbf{f}(\mathbf{x}) = \boldsymbol{J}(\mathbf{x})(\mathbf{x}_A^* - \mathbf{x}) + O(\|\mathbf{x}_A^* - \mathbf{x}\|^2).$$

We *ignore* the quantity $O(\|\mathbf{x}_A^* - \mathbf{x}\|^2)$ because this *ought* to be small and $\|\mathbf{x}_A^* - \mathbf{x}\| \approx C\varepsilon$ for term that involves the machine $\varepsilon \approx 10^{-16}$, so this quantity ought to scale as 10^{-32}. That's small enough to ignore. Then note

$$\frac{\|\mathbf{f}(\mathbf{x}_A^*) - \mathbf{f}(\mathbf{x})\|}{\|\mathbf{f}(\mathbf{x})\|} \text{``}=\text{''} \frac{\|\boldsymbol{J}(\mathbf{x})(\mathbf{x}_A^* - \mathbf{x})\|}{\|\mathbf{f}(\mathbf{x})\|} \le \underbrace{\frac{\|\boldsymbol{J}(\mathbf{x})\|\,\|\mathbf{x}\|}{\|\mathbf{f}(\mathbf{x})\|}}_{\substack{\kappa_f(\mathbf{x}) \\ \text{condition number}}} \quad \underbrace{\frac{\|\mathbf{x}_A^* - \mathbf{x}\|}{\|\mathbf{x}\|}}_{\substack{\text{the algorithm} \\ \text{(below)}}}$$

THE ALGORITHM CONDITION NUMBER

An algorithm is map too. Here we have $\mathbf{f}_A : \mathbb{M}^m \to \mathbb{M}^m$ (machine numbers to machine numbers!).

Simple Example: $f(x_1, \ldots, x_m) = \prod_{i=1}^m x_i$. Here is a simple algorithm.

```
function myprod(x::Vector)
  p = x[1]
  for i=2:length(x)
    p = p*x[i]
  end
  return p
end
```

This corresponds to the mathematical operations.

```
p_1 = x_1
p_2 = fl(p_1 x_1) = p_1 \odot x_1
...
p = fl(p_{m-1} x_m) = p_{m-1} \odot x_m
```

This is the algorithm: p is a machine number! as is p_i.

THE ASSUMPTION

For any computer implementation of $\mathbf{f}$, we assume that

$$\mathbf{f}_A(\mathbf{x}) = \mathbf{f}(\mathbf{x}_A) \text{ for some } \mathbf{x}_A \in \mathbb{R}^m .$$

This is called *backwards stability*.

Key points:

- $\mathbf{x}_A^*$ is in the *reals*
- for a new algorithm, you need to show this is the case
- assume $\mathbf{x}$ is already machine represented.
- The choice of $\mathbf{x}_A$ is not be unique!

For the multiplication algorithm, we have:

$$\begin{aligned} p_1 &= x_1 \\ p_2 &= \mathrm{fl}(x_2 p_1) = x_2 x_1 (1+\varepsilon_1) \\ p_3 &= \mathrm{fl}(x_3 p_2) = x_3 p_2 (1+\varepsilon_2) = x_3 x_2 (1+\varepsilon_2) x_1 (1+\varepsilon_1) \\ &\vdots \\ p_m &= x_m x_{m-1}(1+\varepsilon_{m-1}) \cdots x_2 (1+\varepsilon_2) x_1 (1+\varepsilon_1) \end{aligned}$$

So we have the true product of

$$x_m, \quad x_{m-1}(1+\varepsilon_{m-1}), \quad \ldots, \quad x_1(1+\varepsilon_1)$$

CONDITION NUMBER OF THE ALGORITHM

The condition number of ALG is how far away the input needs to go to satisfy backwards stability. This is the relationship between $\mathbf{x}_A$ and $\mathbf{x}$.

Formally,

$$(\text{cond A})(\mathbf{x}) \quad = \quad \inf_{\mathbf{x}_A} \quad \frac{\|\mathbf{x}_A - \mathbf{x}\|}{\|\mathbf{x}\|} \Big/ u \quad \leq \quad \operatorname*{any}_{\mathbf{x}_A} \quad \frac{\|\mathbf{x}_A - \mathbf{x}\|}{\|\mathbf{x}\|} \Big/ u$$

For the multiplication algorithm we had:

$$\begin{aligned} \mathbf{x} &= \begin{bmatrix} x_1 & x_2 & \ldots & x_m \end{bmatrix} \\ \mathbf{x}_A &= \begin{bmatrix} x_1(1+\varepsilon_1) & x_2(1+\varepsilon_2) & \ldots & x_m \end{bmatrix} \\ \mathbf{x}_A - \mathbf{x} &= \begin{bmatrix} \varepsilon_1 x_1 & \varepsilon_2 x_2 & \ldots, 0 \end{bmatrix} \end{aligned}$$

So we have $\|\mathbf{x}_A - \mathbf{x}\|_\infty \leq u \|\mathbf{x}\|_\infty$, and this gives us $(\text{cond A})(\mathbf{x}) = 1$.

ANOTHER EXAMPLE

$f(x) = \log(x), x \in \mathbb{M}$.

Assume we have an algorithm for log that gives:

$$f_A(x) = \log(x)(1+\varepsilon) \quad |\varepsilon| \le 5u.$$

This implies that

$$f_A(x) = \log(x^{(1+\varepsilon)}),$$

which is the backward stability bound.

$x_A = x^{(1+\varepsilon)}$

$\frac{|x-x_A|}{|x|} = \frac{|x^{(1+\varepsilon)}-x|}{|x|} = |x^{\varepsilon} - 1| = |e^{\varepsilon \log x} - 1| \approx \varepsilon |\log x|$

$(\text{cond A})(x) = 5|\log x|$

1 THE OVERALL ERROR ANALYSIS ON THE COMPUTER

Our idea here is to decouple these two sources of error in our analysis to let us analyze each piece and study them!

Now we need to put all the pieces together.

$$\frac{\|\mathbf{f}(\mathbf{x}_A^*) - \mathbf{f}(\mathbf{x})\|}{\|\mathbf{f}(\mathbf{x})\|} \text{``}=\text{''} \frac{\|\boldsymbol{J}(\mathbf{x})(\mathbf{x}_A^* - \mathbf{x})\|}{\|\mathbf{f}(\mathbf{x})\|} \le \kappa_f(\mathbf{x}) \frac{\|\mathbf{x}_A^* - \mathbf{x}\|}{\|\mathbf{x}\|} \le \kappa_f(\mathbf{x}) \left(\frac{\|\mathbf{x}_A^* - \mathbf{x}^*\|}{\|\mathbf{x}\|} + \frac{\|\mathbf{x}^* - \mathbf{x}\|}{\|\mathbf{x}\|} \right)$$

We generally assume the input error

$$\frac{\|\mathbf{x}^* - \mathbf{x}\|}{\|\mathbf{x}\|} \le \varepsilon.$$

For the next term, we need one more step

$$\frac{\|\mathbf{x}_A^* - \mathbf{x}^*\|}{\|\mathbf{x}\|} = \frac{\|\mathbf{x}_A^* - \mathbf{x}^*\|}{\|\mathbf{x}\|} \frac{\|\mathbf{x}^*\|}{\|\mathbf{x}^*\|} = \frac{\|\mathbf{x}_A^* - \mathbf{x}^*\|}{\|\mathbf{x}^*\|} \frac{\|\mathbf{x}^*\|}{\|\mathbf{x}\|}$$

Now note that $\frac{\|\mathbf{x}^*\|}{\|\mathbf{x}\|} \approx (1+\varepsilon)$.