REPRESENTING FUNCTIONS ON COMPUTERS

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February 9, 2021

These notes are based on Chapter 2 in Gautschi's Numerical Analysis textbook.

THE QUESTION OF CHAPTER 1

How do we represent numbers on a computer and how does this representation impact what we might want to compute?

THE QUESTION OF CHAPTER 2

How do we represent mathematical functions on a computer?

The following may seem like a silly question, but what *is* a function f on a computer?

- Is it?

```
function f(x)
return sin(100x)
end
```

- Is it?

```
f(x) = 100x - 500000 x^3 / 3 + 250000000 x^5 / 3
```

- Is it?

```
pushq   %rax
imulq   100, %rdi, %rax
vcvtsi2sd   %rax, %xmm0, %xmm0
movabsq sin, %rax
callq   *%rax
popq    %rax
retq
```

- Is it a picture of f?

We assume we have *oracle access* to f and we can compute $f(x)$ for some input.

LINEAR FUNCTIONS SPACES

The key way we work with functions on the computer are through function spaces. These allow us to represent functions as small sets of numbers. Just as floating point values limit what we can do with respect to real numbers, these functions spaces will also be limited.

A linear function space Φ is an infinite set of functions from $\mathbb{R}$ to $\mathbb{R}$ where

- $f_1, f_2 \in \Phi \implies f_1 + f_2 \in \Phi$, and
- $cf_1 \in \Phi$ for all $c \in \mathbb{R}$.

Example The set $\Phi = \{f(x) = ax + b \mid a, b \in \mathbb{R}\}$ is a linear function space. This just represents all line functions, so maybe this isn't surprising at all. To check this ...

TODO in notes, check that multiplying by c gives us something else there. Check that addition gives us something else in the set too.

The key point is that we can represent an *element* in this infinite function space by *two numbers*: a, b.

OVERVIEW AND APPROXIMATION THEORY

To discuss how to represent functions on a computer, we consider *best approximation problems*.

This will involve comparing functions, for which, we need *function norms* and more exotic things like weighted functions norms.

THE BEST APPROXIMATION PROBLEM FOR FUNCTIONS

We have a linear function space $\Phi = \underbrace{\{\phi_i\}}_{\text{infinite set of functions}}$.

Given a function f, find ϕ such that $\phi \in \Phi$ and f is as close to ϕ as possible.

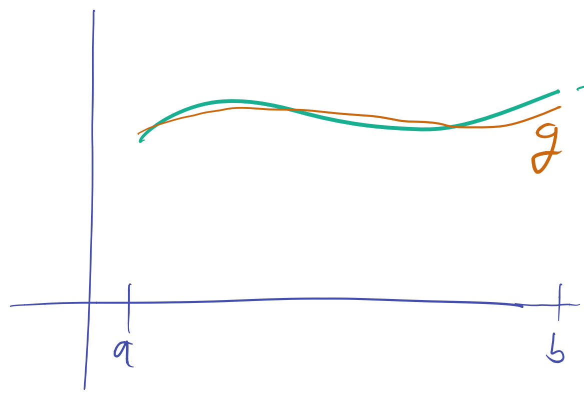
To do this requires a measure of distance between functions. This is a *function norm*.

Note that this is very close to the chapter 1 problem of given $x \in RR$, find closest $x^* \in \mathbb{M}$.

AN INITIAL IDEA FOR FUNCTION NORMS

Given f and g, and a region $[a, b]$, how close are they? Well, first thought, just *look* at f and g!

Sometimes this is called the *eyeball norm!* $\|\mathbf{f} - \mathbf{g}\|_{👁}$



Of course, the computer can't actually look at the pictures of the functions. So here is a simple computational alternative.

Pick n points in $[a, b]$, evaluate f and g at those points. Let $x_1, \ldots, x_n$ be the points. If we assemble these measurements into vectors, then we can simply compare the vectors!

$$\mathbf{u} = \begin{bmatrix} f(x_1) \\ f(x_2) \\ \vdots \\ f(x_n) \end{bmatrix} \qquad \mathbf{v} = \begin{bmatrix} g(x_1) \\ g(x_2) \\ \vdots \\ g(x_n) \end{bmatrix}$$

Then $\|\mathbf{u} - \mathbf{v}\|$ is a simple measurement of distance between f and g.

There are issues with this idea, though. For instance, what points do we pick? How do the distances change as we vary the number of points?

FUNCTION NORMS

Recall that, if we used the infinity norm, then

$$\|\mathbf{u} - \mathbf{v}\|_\infty = \max_i |f(x_i) - g(x_i)|.$$

If we take $n \to \infty$, then we simply look at all points between a and b.

DEFINITION 1 *The infinity norm of a function is*

$$\|f\|_\infty = \max_{a \le x \le b} |f(x)|.$$

The infinity norm distance between functions f and g is

$$\|f-g\|_\infty = \max_{a\le x\le b} |f(x)-g(x)|.$$

This same idea also motivates the following norms

DEFINITION 2 *The 2-norm of a function is*

$$\|f\|_2 = \sqrt{\int_a^b |f(x)|^2\,dx}.$$

The 2-norm distance between functions f and g is

$$\|f-g\|_2 = \sqrt{\int_a^b |f(x)-g(x)|^2\,dx}.$$

DEFINITION 3 *The 1-norm of a function is*

$$\|f\|_1 = \int_a^b |f(x)|\,dx.$$

The 1-norm distance between functions f and g is

$$\|f-g\|_1 = \int_a^b |f(x)-g(x)|\,dx.$$

SIMPLE EXAMPLES

These allow us to address simple questions. Is $f(x) = x$ a good approximation to $g(x) = x^2$ over a region around 1?

Let $[1-h, 1+h]$ quantify the region around 1, then we are interested in

$$\|x^2 - x\|_2 = \sqrt{\int_{1-h}^{1+h} (x^2-x)^2\,dx} = \ldots \text{ wolfram alpha } \ldots = \sqrt{\frac{2}{5}h^5 + \frac{2}{3}h^3}.$$