REPRESENTING FUNCTIONS ON COMPUTERS

David F. Gleich

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These notes are based on Chapter 2 in Gautschi's Numerical Analysis textbook.

In this lecture, we look at how we find the *best* computer approximation of a function. This will give us computational representations.

THE PROBLEM SETUP

Given a function f, find $\phi \in Phi$ a linear function space.

What is f? f is simply what you are interested in working with! It could be something simple, like $\sin(x)$. Or it could be something complicated, like the amount of fuel used by a SpaceX rocket moving a satellite to a geostationary transfer orbit. That second function is much more complicated and I hope you agree it's possible to *simulate* the function for particular points in geostationary transfer orbit, but having any insight into the function "as a math expression" is hard.

A simple setting for linear functions spaces is as a linear combination of elementary functions. These are nice because we can represent elements in these function spaces with their coefficients

Example. Let $\pi_1(t), \ldots, \pi_n(t)$ be linearly independent functions. Then $\Phi = \{c_1\pi_1(t) + c_2\pi_2(t) + \ldots + c_n\pi_n(t) \mid c_i \in \mathbb{R}\}$

Linearly independent functions. (Or ?) In linear algebra with vectors, we have

$$\mathbf{v}_1, \ldots, \mathbf{v}_k$$

are linearly independent, then $c_1\mathbf{v}_1 + \ldots + c_k\mathbf{v}_k = 0$ is true if and only if $c_1, \ldots, c_k = 0$. So the same definition also holds for functions A linearly independent set of functions is characterized by the property that the only way of representing the zero function $f(t) = 0$ is where all the scalar terms $c_1, \ldots, c_k$ are zero.

LIMITING OUR CLASS OF FUNCTIONS

Now, the above setting is very general. We often look at standard classes.

Two standard classes are

- polynomials for general problems
- trig functions for periodic domains

Why these two classes? Well, let's address polynomials first.

Polynomials are fairly easy to work with and reason about. They are linear combinations of monomials.[1]

[1] Although this is a terrible way to work with them! Do not use monomial polynomial representations unless you have very small degree polynomials.

The second reason polynomials are helpful is the Weierstrauss approximation theorem.

WEIERSTRAUSS APPROXIMATION THEOREM

See Chapter 6 in Trefethen ATAP for more on this theorem.

Let f be a continuous function on $[-1, 1]$. Fix $\varepsilon > 0$. Then there exists a polynomial p such that

$$\|f - p\|_\infty \leq \varepsilon.$$

The proof is hard and intricate and not particularly relevant. But it relies on *morphing* between a simple function $\hat{f}$ and the original continuous function f using a PDE. This is a heat flow PDE like you see in morphing animations. As we reverse this process, it's smooth (because it's a heat flow PDE) and super continuous. Each function at each step then has a convergence taylor approximation, i.e. a polynomial.

SOLVING BEST APPROXIMATION PROBLEMS AND THE NORMAL EQUATIONS

Let Φ be linear function space with basis function π_i and coefficients c_i. Our goal is to find $\phi \in \Phi$, or equivalently, c_i, to represent an arbitrary function f.

Consequently, the least squares approximation is find $\phi \in Phi$ where

$$\|f - \phi\|_2$$

is as small as possible.

Assume that the π_i are linearly independent.

Then we have

$$E = \|f - \phi\|_2^2 = \left\|f - \sum_i c_i \pi_i\right\|_2^2 \ldots \text{ lots of algebra in class}$$

This can be written in terms of terms involving $\int \pi_i \pi_j \, dx$ or $\int f \pi_i \, dx$.

These are something called *inner products*.

ASIDE ON INNER PRODUCTS

The inner-product of two functions f and g is the result

$$\int_a^b f(x) g(x) \, dx \quad \text{or equivalently} \quad \int f(t) g(t) \, d\lambda(t) \quad \text{or equivalently} \quad \int f g \, d\lambda(t)$$

$$\text{or equivalently} \quad (f, g)$$

These inner product have some nice properties

- Symmetric $(f, g) = (g, f)$
- Homogeneous $(\alpha f, g) = \alpha(f, g)$
- Additive $(u + v, w) = (u, w) + (v, w)$
- Positive Definite $(u, u) \geq 0$ and $(u, u) = 0$ if and only if $u = 0$.

These, of course, are really the properties of linearity.