In this class you should learn:

- *Why the derivative problem is fundamentally ill-conditioned*
- *Derivatives of polynomial approximations*
- *High dimensional polynomials*

November 8, 2016

HW Due, Odds & Ends of important topics

Next class

Numerical integration

Next next class

QUIZ!
More numerical integration!

Ill conditioning & numerical differentiation

Juliabox Demo!

The take-away intuition

Numerical differentiation is sensitive to errors in the function values.

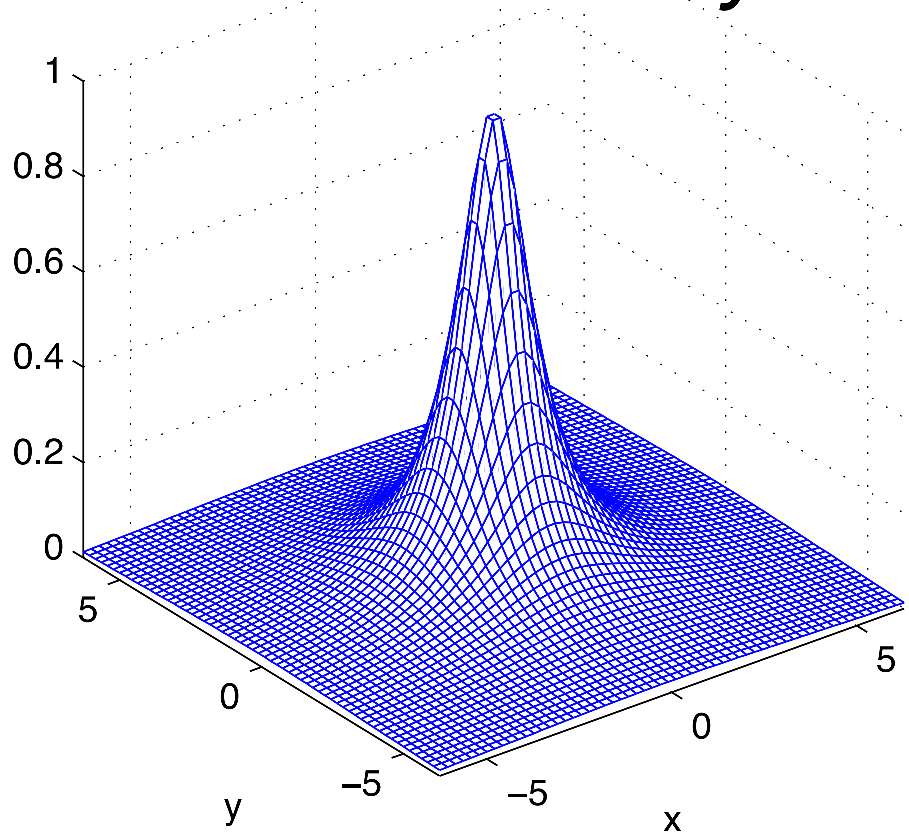
But the sensitivity seems mostly proportional to the magnitude of the perturbation.

It doesn’t grow “exponentially”

Not especially ill-conditioned away from singularities

Multivariate functions

$$f(x, y) = \frac{1}{1 + x^2 + y^2}$$



Multivariate polynomials

$$
\begin{aligned}
f(x, y) = \; & c_{2,2}x^2y^2 && + \; c_{1,2}xy^2 && + \; c_{0,2}y^2 && + \\
& c_{2,1}x^2y && + \; c_{1,1}xy && + \; c_{0,1}y && + \\
& c_{2,0}x^2 && + \; c_{1,0}x && + \; c_{0,0} &&
\end{aligned}
$$

A bi-variate (2 variable) quadratic has 9 unknown parameters

A bi-variate (2 variable) cubic has 16 unknown parameters

A tri-variate (3 variable) quadratic has 27 unknown parameters

A tri-variate (3 variable) cubic has ? unknown parameters

Degree of multivariate polynomials

the degree of a multivar poly is the degree of the largest term

Degree of multivariate polynomials

$x^2 y^2$ has degree "four" $x y^2$ has degree "three"

the degree of a multivar poly is the degree of the largest term

Degree of multivariate polynomials

$$\begin{aligned} f(x, y) = \; & c_{2,2}x^2y^2 && + \quad c_{1,2}xy^2 && + \quad c_{0,2}y^2 && + \\ & c_{2,1}x^2y && + \quad c_{1,1}xy && + \quad c_{0,1}y && + \\ & c_{2,0}x^2 && + \quad c_{1,0}x && + \quad c_{0,0} \end{aligned}$$

x^2 y^2 has degree "four" x y^2 has degree "three"

the degree of a multivar poly is the degree of the largest term

Degree of multivariate polynomials

$$
\begin{aligned}
f(x, y) = \quad & c_{2,2}x^2y^2 && + && c_{1,2}xy^2 && + && c_{0,2}y^2 && + \\
& c_{2,1}x^2y && + && c_{1,1}xy && + && c_{0,1}y && + \\
& c_{2,0}x^2 && + && c_{1,0}x && + && c_{0,0}
\end{aligned}
$$

$x^2\, y^2$ has degree "four" $\qquad$ $x\, y^2$ has degree "three"

the degree of a multivar poly is the degree of the largest term

$$
\begin{aligned}
f(x, y) = \quad & && && && && c_{0,2}y^2 && + \\
& && + && c_{1,1}xy && + && c_{0,1}y && + \\
& c_{2,0}x^2 && + && c_{1,0}x && + && c_{0,0}
\end{aligned}
$$

Quiz

Write down the equations for a multi-linear function in three dimensions:

(1) where all degrees are less than or equal to 1

(2) where all “linear” terms of present

$$
\begin{aligned}
f(x,y) = \;& c_{2,2}x^2y^2 &&+ c_{1,2}xy^2 &&+ c_{0,2}y^2 &&+ \\
& c_{2,1}x^2y &&+ c_{1,1}xy &&+ c_{0,1}y &&+ \\
& c_{2,0}x^2 &&+ c_{1,0}x &&+ c_{0,0} &&
\end{aligned}
$$

$$
\begin{aligned}
f(x,y) = \;& && && c_{0,2}y^2 &&+ \\
& &&+ c_{1,1}xy &&+ c_{0,1}y &&+ \\
& c_{2,0}x^2 &&+ c_{1,0}x &&+ c_{0,0} &&
\end{aligned}
$$

Fitting multivariate polynomials

$$
\begin{array}{lccccccc}
f(x,y) = & & & & & c_{0,2}y^2 & + \\
& & + & c_{1,1}xy & + & c_{0,1}y & + \\
& c_{2,0}x^2 & + & c_{1,0}x & + & c_{0,0} &
\end{array}
$$

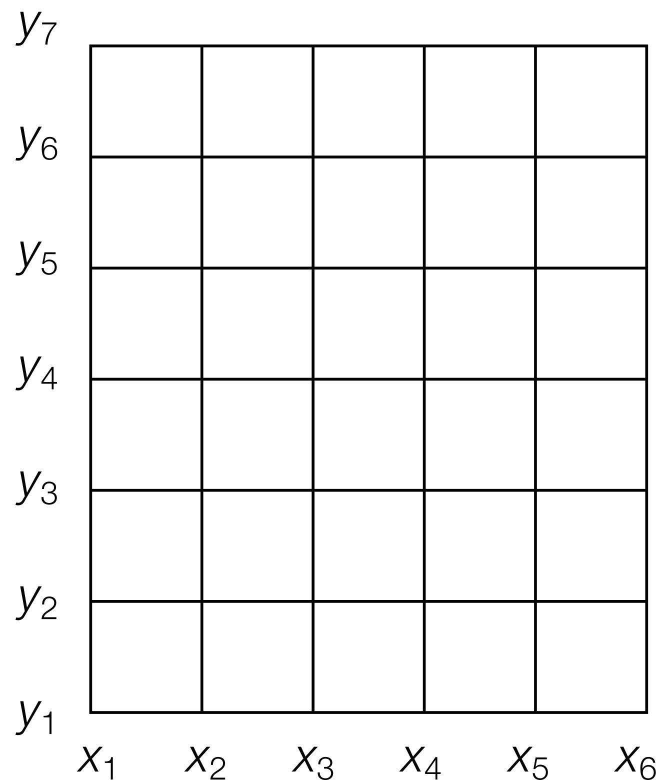
… not nice to write down in general …

Saniee. "A simple form of the multivariate Lagrange interpolant"
SIAM J. Undergraduate Research Online, 2007.

An easier special case

If we have data from our polynomial on a repeated grid then we can fit a sum of 1d polynomials

"tensor product constructions"



$$p(x, y) = \sum z_{ij} \varphi_i^x(x) \varphi_j^y(y)$$

The big problem

If we have an m dimensional function

And we want an n degree interpolant

We need $(n+1)^m$ samples of our function.

“quadratic” in 10 dimensions – 3^{10} samples

“quadratic” in 100 dimensions – 3^{100} samples

Exponential growth or “curse of dimensionality”