In this class you should learn:

- A new type of error that arises in numeric differentiation called truncation error

- And how truncation error and floating point error need to be balanced for accurate computations

- What “high dimensional polynomials” require and why they make it hard.
Piecewise polynomial approximation

Used all over!
Powerpoint curves
Piecewise polynomial approximation
Piecewise polynomial approximation
\[ \ell(x) = f(x_{i-1}) \frac{x - x_i}{x_{i-1} - x_i} + f(x_i) \frac{x - x_{i-1}}{x_i - x_{i-1}} \]

In each subinterval
In each subinterval

\[ \ell(x) = f(x_{i-1}) \frac{x - x_i}{x_{i-1} - x_i} + f(x_i) \frac{x - x_{i-1}}{x_i - x_{i-1}} \]
Piecewise polynomial approximation

**Linear** Uses a set of functions values & points
   Good if there are *many* points

**Quadratic** Same info
   Can use extra point to match midpoints

**Cubic Hermite** Uses points, function values, and derivatives
   Matches the function values and derivatives!

**Cubic Splines** Uses points, function values
   A twice continuously differentiable interpolant!
Piecewise polynomial approximation

**Linear**

\[ \ell(x) = f(x_{i-1}) \frac{x - x_i}{x_{i-1} - x_i} + f(x_i) \frac{x - x_{i-1}}{x_i - x_{i-1}} \]

Easy!

**Quadratic**

Easy!

**Cubic Hermite**

Local work

**Cubic Splines**

Global work
4 parameters, 4 unknowns in the cubic polynomial between $x_i, x_{i+1}$

Fit via differentiation.

One continuous derivative! See the book.
Cubic Splines

4(n-1) parameters, e.g. 4 unknowns in the cubic polynomial between $x_i$, $x_{i+1}$

Matching points gives 2(n-1) constraints

Derivatives are continuous (n-3) constraints

2nd derivatives are continuous (n-3) constraints
Cubic Splines

\[ s''_i(x) = z_{i-1} \frac{x - x_i}{x_{i-1} - x_i} + z_i \frac{x - x_{i-1}}{x_i - x_{i-1}} \]

Piecewise linear second derivative

Continuous derivative gives us a linear system

\[
\begin{bmatrix}
\alpha_1 & \beta_1 \\
\beta_1 & \alpha_2 & \beta_2 \\
\beta_2 & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots \\
\beta_{n-2} & \ddots & \ddots & \ddots & \beta_{n-2} \\
\beta_{n-2} & \ddots & \ddots & \ddots & \alpha_{n-1} \\
\end{bmatrix}
\begin{bmatrix}
z_1 \\
z_2 \\
\vdots \\
z_{n-1} \\
\end{bmatrix}
= 
\begin{bmatrix}
b_1 \\
b_2 \\
\vdots \\
b_{n-1} \\
\end{bmatrix}
\]
Multivariate functions

\[ f(x, y) = \frac{1}{1 + x^2 + y^2} \]
Multivariate polynomials

\[ f(x, y) = c_{2,2}x^2y^2 + c_{1,2}xy^2 + c_{0,2}y^2 + c_{2,1}x^2y + c_{1,1}xy + c_{0,1}y + c_{2,0}x^2 + c_{1,0}x + c_{0,0} \]

A bi-variate (2 variable) quadratic has 9 unknown parameters
A bi-variate (2 variable) cubic has 16 unknown parameters

A tri-variate (3 variable) quadratic has 27 unknown parameters
A tri-variate (3 variable) cubic has ? unknown parameters
Degree of multivariate polynomials

\[ f(x, y) = c_{2,2}x^2y^2 + c_{1,2}xy^2 + c_{0,2}y^2 + c_{2,1}x^2y + c_{1,1}xy + c_{0,1}y + c_{2,0}x^2 + c_{1,0}x + c_{0,0} \]

\( x^2y^2 \) has degree “four”  \( xy^2 \) has degree “three”

the degree of a multivar poly is the degree of the largest term

\[ f(x, y) = c_{0,2}y^2 + c_{1,1}xy + c_{0,1}y + c_{2,0}x^2 + c_{1,0}x + c_{0,0} \]
Write down the equations for a multi-linear function in three dimensions:

(1) where all degrees are less than or equal to 1

(2) where all “linear” terms of present

\[ f(x, y) = c_{2,2}x^2y^2 + c_{1,2}xy^2 + c_{0,2}y^2 + c_{2,1}x^2y + c_{1,1}xy + c_{0,1}y + c_{2,0}x^2 + c_{1,0}x + c_{0,0} \]

\[ f(x, y) = c_{0,2}y^2 + c_{1,1}xy + c_{0,1}y + c_{2,0}x^2 + c_{1,0}x + c_{0,0} \]
Fitting multivariate polynomials

\[ f(x, y) = c_{0,2}y^2 + c_{1,1}xy + c_{0,1}y + c_{2,0}x^2 + c_{1,0}x + c_{0,0} \]

… not nice to write down in general …

An easier special case

If we have data from our polynomial on a repeated grid then we can fit a sum of 1d polynomials

“tensor product constructions”

\[ p(x, y) = \sum z_{ij} \varphi_i^x(x) \varphi_j^y(y) \]
The big problem

If we have an $m$ dimensional function
And we want an $n$ degree interpolant

We need $(n+1)^m$ samples of our function.

“quadratic” in 10 dimensions – $3^{10}$ samples
“quadratic” in 100 dimensions – $3^{100}$ samples
Exponential growth or “curse of dimensionality”
Quiz

Write down the equations for a linear interpolant in three dimensions:

(1) where all degrees are less than 1

(2) where all "linear" terms of present
Chebfun implements 1d polynomial interpolation, just use it!
Equally spaced points are bad for interpolation!
There are many mathematically equivalent variations on unique polynomial interpolants but they have different computation properties
Piece-wise polynomials are a highly-flexible model for modeling general smooth curves.
High dimensional interpolation is really hard