INTRODUCTION TO ORDINARY DIFFERENTIAL EQUATIONS

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1 EXAMPLES

Malthusian population growth (Isn't this one awesome term? Malthusian!) This is the idea that population grows at a rate proportional to the current population. So if $y(t)$ is the population at time t, then

$$\frac{dy}{dt} = Cy(t).$$

This gives the exponential solution from initial population $y(0)$

$$y(t) = y(0)e^{Ct}.$$

Hooke's law

$$\frac{d^2y}{dt^2} = -ky.$$

Rabbits & Foxes Or the Lotka-Volterra system.

2 GENERAL FORM

The general form of the problem we'll work with is:

$$\frac{dy}{dt} = f(t, y)$$

$$y' = f(t, y)$$

But this also holds for systems:

$$\begin{bmatrix} \frac{dy}{dt} \\ \frac{dz}{dt} \end{bmatrix} = \begin{bmatrix} f(t, y, z) \\ g(t, y, z) \end{bmatrix}$$

For these, we usually just write:

$$\frac{d\mathbf{y}}{dt} = \mathbf{f}(t, \mathbf{y})$$

where we treat $\mathbf{y}(t)$ as a vector of d terms, and $\mathbf{f}$ as a vector of d terms.

Malthusian $d = 1$ (a scalar problem) and

$$f(t, y) = Cy.$$

Hooke's law $d = 2$ (a system problem) and

$$f(t, \mathbf{y}) = \begin{bmatrix} 0 & 1 \\ -k & 1 \end{bmatrix}\mathbf{y}, \text{ where } \mathbf{y} = \begin{bmatrix} \text{position}(t) \\ \text{velocity}(t) \end{bmatrix}.$$

Rabbits and Foxes

3 PROPERTIES OF ODES

For our rigorous analysis of ODEs to work, we need a few assumptions: A ***Well-posed*** problem and a ***Unique*** solution. Your book has technical definitions, here, let's use:

Well-posed small changes don't change things.

Unique starting from $\mathbf{y}(0)$ there is only one solution.

4 Our first method

In general

$$\frac{dy}{dt} = f(t, y) \quad y(0) = s$$

An example

5 Definitions

Consider a one-step method: $y_{k+1} = y_t + h\mathtt{step}(y_t, t, h)$. The `step` stuff is what you would compute in your method, for instance, in forward Euler, $\mathtt{step}(y, t, h) = f(t, y)$.

A method is ***consistent*** if $\lim_{h\to 0} \mathtt{step}(y, t, h) = f(t, y)$. So that means that if we took an infinitesimally small step, we'd do the right thing.

Let $y(t)$ solve the ODE. The ***local truncation error*** is the difference:

$$\frac{y(t+h) - y(t)}{h} - \mathrm{step}(y(t), t, h).$$

This can be interpreted as *how much does step differ from the slope of the true solution?*

A one step method is ***stable*** if there is a constant K and a step size h_0 such that the difference between two solutions $y_n, \tilde{y}_n$ started from $y_0 - \tilde{y}_0$

$$|y_n - \tilde{y}_n| \leq K|y_0 - \tilde{y}_0|.$$

This notion of stability isn't very important for us. Almost all of the methods we'll look at are stable in this sense, and the book proves that any one-step method of the form above is stable for an ODE with a *Lipschitz* function $f(t, h)$.

THEOREM 1 (11.2.2) *If the method $y_{k+1} = y_t + h\mathtt{step}(y_t, t, h)$ is stable and consistent, with local truncation error $O(h^p)$, then the global error is $O(h^p)$.*

6 Lipschitz?

A function is called Lipschitz continuous if $|f(x) - f(y)| \leq L|x - y|$ for all x, y. This is a SUPER strong notion of continuous. If you haven't seen it before, don't worry about it – it just means *continuous enough for small changes not to become a problem*.

7 Julia's ODE suite

8 STIFF EQUATIONS

Chemical Kinetics (From Leveque, Finite Difference Methods, Chapter 7)
Imagine yourself as a chemical engineering. We need to produce the molecule AB from the constituent molecules A and B. This occurs with rate K_1. But AB also breaks down into A and B with rate K_2. Can you get a solution of 99%AB only? (Assume we can freeze the solution to stop both reactions.)

The mathematical model

Forward Euler solution

u_1 = concentration of A
u_2 = concentration of B
u_3 = concentration of AB

$$u_1' = -K_1 u_1 u_2 + K_2 u_3$$
$$u_2' = -K_1 u_1 u_2 + K_2 u_3$$
$$u_3' = K_1 u_1 u_2 - K_2 u_3$$

If $K_1 \gg K_2$, then we can get good purity, but it may be hard to accurately simulate the ODEs. This is the hallmark of *stiff equations*, those with vastly different "time scales"

9 ABSOLUTE STABILITY

Test equation 1 $y' = \lambda y$

Test equation 2 $\mathbf{y}' = \mathbf{A}\mathbf{y}$

If $\lambda < 0$, then $y \to 0$ as $t \to \infty$

If *eigenvalues of **A*** have *real part* < 0, then $\mathbf{y} \to 0$ as $t \to \infty$

Consider a one-step method $y_{k+1} = y_t + h\texttt{step}(\mathbf{y}_t, t, h)$ for $y' = \lambda y$.
Let λ be real or complex.
The ***region of absolute stability*** is the set $\{h\lambda\}$ where $y_k \to 0$ as $k \to \infty$.

10 BACKWARD EULER

11 LINGO

ODE
ordinary differential equation, just one type of derivative, usually in a variable called time

BVP
boundary value problem, an ODE with one, or multiple end-points fixed. (think "starting at time t_0") as one end-point of the time grid.

PDE
partial differential equation, usually involves derivatives of space and time.

time grid, time mesh
the set of times where we have an approximate solution of an ODE or PDE

scheme
a method to *evaluate* or *integrate* an ODE, e.g. $y_{k+1} = y_k + \texttt{step}(y_k, t, h)$.

integrator
the name for an overarching approach or function that, given an initial condition of an ODE, produces an approximate solution.

sympletic integrator
A very special type of integrator for systems of equations that are designed for systems of motion like the spring system. These systems are designed to preserve energy over time.

multi-step methods
A method that uses the solution from multiple points in time. We don't discuss these.

Runge-Kutta
A class of high-order methods for integrating ODEs that only use one-step. The RK45 method is one of the most standard choices.

implicit
a scheme where the function evaluation depends *implicitly* on the solution at the next time step. Implicit schemes require solving a system of equations (possibly nonlinear)