1 Examples

**Malthusian population growth** (Isn't this one awesome term? Malthusian!) This is the idea that population grows at a rate proportional to the current population. So if $y(t)$ is the population at time $t$, then

$$\frac{dy}{dt} = Cy(t).$$

This gives the exponential solution from initial population $y(0)$

$$y(t) = y(0)e^{Ct}.$$  

**Hooke's law**

$$\frac{d^2y}{dt^2} = -ky.$$  

**Rabbits & Foxes** Or the Lotka-Volterra system.

2 General Form

The general form of the problem we'll work with is:

$$\frac{dy}{dt} = f(t, y)$$

But this also holds for systems:

$$\begin{bmatrix} \frac{dy}{dt} \\ \frac{dz}{dt} \end{bmatrix} = \begin{bmatrix} f(t, y, z) \\ g(t, y, z) \end{bmatrix}$$

For these, we usually just write:

$$\frac{dy}{dt} = f(t, y)$$

where we treat $y(t)$ as a vector of $d$ terms, and $f$ as a vector of $d$ terms.

**Malthusian** $d = 1$ (a scalar problem) and

$$f(t, y) = Cy.$$  

**Hooke's law** $d = 2$ (a system problem) and

$$f(t, y) = \begin{bmatrix} 0 & 1 \\ -k & 1 \end{bmatrix} y,$$ where $y = \begin{bmatrix} \text{position}(t) \\ \text{velocity}(t) \end{bmatrix}.$

**Rabbits and Foxes**

3 Properties of ODEs

For our rigorous analysis of ODEs to work, we need a few assumptions: A *Well-posed* problem and a *Unique* solution. Your book has technical definitions, here, let's use:

**Well-posed** small changes don't change things.

**Unique** starting from $y(0)$ there is only one solution.
4 Our first method

In general
\[ \frac{dy}{dt} = f(t, y) \quad y(0) = s \]

An example

Consider a one-step method: \( y_{k+1} = y_t + h \text{step}(y_t, t, h) \). The step stuff is what you would compute in your method, for instance, in forward Euler, \( \text{step}(y, t, h) = f(t, y) \).

A method is consistent if \( \lim_{h \to 0} \text{step}(y, t, h) = f(t, y) \). So that means that if we took an infinitesimally small step, we'd do the right thing.

Let \( y(t) \) solve the ODE. The local truncation error is the difference:
\[ \frac{y(t+h) - y(t)}{h} - \text{step}(y(t), t, h) \]

This can be interpreted as how much does step differ from the slope of the true solution?

A one step method is stable if there is a constant \( K \) and a step size \( h_0 \) such that the difference between two solutions \( y_n, \tilde{y}_n \) started from \( y_0 - \tilde{y}_0 \)
\[ |y_n - \tilde{y}_n| \leq K|y_0 - \tilde{y}_0| \]

This notion of stability isn't very important for us. Almost all of the methods we'll look at are stable in this sense, and the book proves that any one-step method of the form above is stable for an ODE with a Lipschitz function \( f(t, h) \).

**THEOREM 1** (11.2.2) If the method \( y_{k+1} = y_t + h \text{step}(y_t, t, h) \) is stable and consistent, with local truncation error \( O(h^p) \), then the global error is \( O(h^p) \).

5 Definitions

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6 Lipschitz?

A function is called Lipschitz continuous if \( |f(x) - f(y)| \leq L|x - y| \) for all \( x, y \). This is a SUPER strong notion of continuous. If you haven't seen it before, don't worry about it – it just means continuous enough for small changes not to become a problem.

7 Julia's ODE Suite
8 STIFF EQUATIONS

Chemical Kinetics (From Leveque, Finite Difference Methods, Chapter 7)
Imagine yourself as a chemical engineer. We need to produce the molecule $AB$ from the constituent molecules $A$ and $B$. This occurs with rate $K_1$. But $AB$ also breaks down into $A$ and $B$ with rate $K_2$. Can you get a solution of 99% $AB$ only? (Assume we can freeze the solution to stop both reactions.)

**The mathematical model**

$$u_1 = \text{concentration of } A$$
$$u_2 = \text{concentration of } B$$
$$u_3 = \text{concentration of } AB$$

$$u_1' = -K_1 u_1 u_2 + K_2 u_3$$
$$u_2' = -K_1 u_1 u_2 + K_2 u_3$$
$$u_3' = K_1 u_1 u_2 - K_2 u_3$$

If $K_1 \gg K_2$, then we can get good purity, but it may be hard to accurately simulate the ODEs. This is the hallmark of **stiff equations**, those with vastly different “time scales”

9 ABSOLUTE STABILITY

**Test equation 1** $y' = \lambda y$

**Test equation 2** $y' = Ay$

If $\lambda < 0$, then $y \to 0$ as $t \to \infty$

If eigenvalues of $A$ have real part $< 0$, then $y \to 0$ as $t \to \infty$

Consider a one-step method $y_{k+1} = y_k + \text{step}(y_k, t, h)$ for $y' = \lambda y$. Let $\lambda$ be real or complex.

The **region of absolute stability** is the set $\{ h \lambda \}$ where $y_k \to 0$ as $k \to \infty$. 
ODE
ordinary differential equation, just one type of derivative, usually in a variable called time

BVP
boundary value problem, an ODE with one, or multiple end-points fixed. (think "starting at time $t_0$") as one end-point of the time grid.

PDE
partial differential equation, usually involves derivatives of space and time.

time grid, time mesh
the set of times where we have an approximate solution of an ODE or PDE

scheme
a method to evaluate or integrate an ODE, e.g. $y_{k+1} = y_k + \text{step}(y_k, t, h)$.

integrator
the name for an overarching approach or function that, given an initial condition of an ODE, produces an approximate solution.

symplectic integrator
A very special type of integrator for systems of equations that are designed for systems of motion like the spring system. These systems are designed to preserve energy over time.

multi-step methods
A method that uses the solution from multiple points in time. We don't discuss these.

Runge-Kutta
A class of high-order methods for integrating ODEs that only use one-step. The RK45 method is one of the most standard choices.

implicit
a scheme where the function evaluation depends implicitly on the solution at the next time step. Implicit schemes require solving a system of equations (possibly nonlinear)