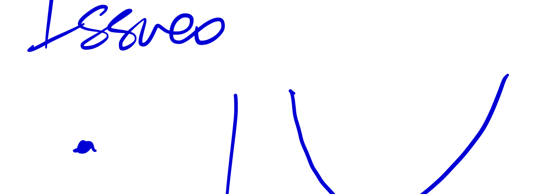
Lecture 40

Non linear Eq:

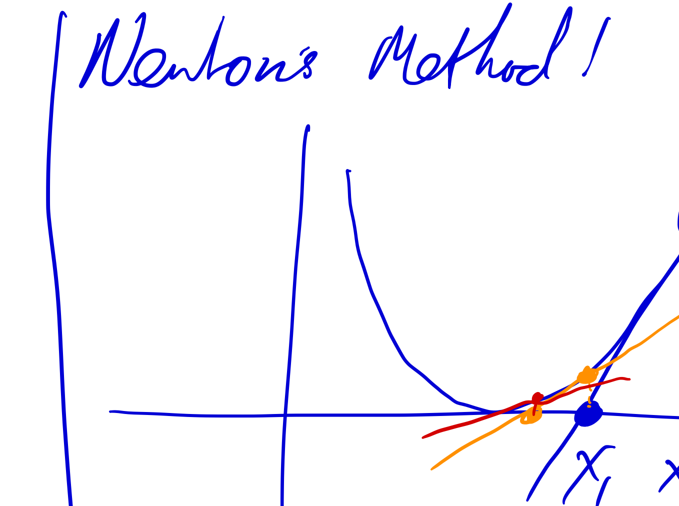
$$f(x) = 0$$

Bisection (last time)

Issues

- 
- Roots of $f(\underline{x}) : \mathbb{R}^m \to \mathbb{R}^n$
  (Multivar case!)
  ↳ No Easy generalization ☹

Newton's Method!



Starts @ a point x_0
↳ form linear approx @ x_0
↳ find zero/root of linear approx. → call x_1

Repeat @ x_1
↳ find linear approx
↳ find root. → x_2

What is linear approx?

$$l(x) = f'(x_0)(x - x_0) + f(x_0)$$

Solve $l(x) = 0$

$$\iff x = x_0 - \frac{f(x_0)}{f'(x_0)}$$

x_1

$$l(x) = f'(x_1)(x - x_1) + f(x_1)$$

$$x = x_1 - \frac{f(x_1)}{f'(x_1)}$$

Method: Subroutine for $f(x)$, $f'(x)$
Then

$$\boxed{x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}}$$

An analytic derivation

Taylor thm:

$$f(x+h) = f(x) + h f'(x) + \frac{h^2}{2} f''(\xi)$$

Ignore $h^2/2$ term! (Approx at ≈ 0)

$f(x+h) \approx f(x) + h f'(x)$ solve for h s.t. $f(x+h) = 0$.

$$f(x+h) \approx f(x) + h f'(x) = 0 \iff$$

$$h = \frac{-f(x)}{f'(x)}$$

So $x + h = x - \frac{f(x)}{f'(x)}$ is an approx root.

Newton's Method does not always work ☹

But if you are close enough it does! ☺

Thm 4.3.1: if x_0 is sufficiently close (c... in terms of ξ) and x^* is the root, then

$$x_k \to x^*$$

(so it converges) and it does so "really fast"
↳ quadratically fast

Quadratic Convergence

$$x_k \to 0$$

$x_1 = 0.1$	1
$x_2 = 0.01$	2
$x_3 = 0.0001$	4
$x_4 = 0.00000001$	8
	16
	32
	64

~~Else~~ Informal @ each step: double the # of correct digits!

Ex:

Compute $\frac{1}{\pi}$ on a ~~Broken~~ calculator w/ keys
$\oplus, \ominus, \otimes, (\pi), (1), (2)$

Look @ $f(x) = 0$ when $x = \frac{1}{\pi}$

and

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

doesn't involve division.

guess 1:

$f(x) = \pi x - 1 = 0$
$f'(x) = \pi$

$$x_{k+1} = x_k - \frac{(\pi x_k - 1)}{\pi} = \frac{1}{\pi} \quad ☹$$

guess 2:

$$f(x) = \frac{1}{x} - \pi$$

$$f'(x) = \frac{-1}{x^2}$$

$$x_{k+1} = x_k - \frac{\left(\frac{1}{x} - \pi\right)}{\left(\frac{-1}{x^2}\right)}$$

$$\boxed{x_{k+1} = x_k(2 - \pi x_k)}$$

Example ②

$f(x) = x^2 - a$ to get $\sqrt{a}$ w/o a ($\sqrt{\ }$) key.

Pros
- fast
- intuitive

Cons
- may not work
- x_0 close.
- derivative (non zero)

Secant: Newton w/ Approx Derivative!

$$f'(x) \approx \frac{f(x+h) - f(x)}{h}$$

In secant, we use the sequence $x_1, x_2, \ldots x_k$ to approx the derivative.

$$f'(x_k) \approx \frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}}$$

⊕ Converge fast (but not as fast as Newton)

⊖ Converge "differently" than newton

Fixed point form of non-linear Eq.

So far $f(x) = 0$

Fixed pt form $g(x) = x$

Why fixed pt? It gives an Alg

$$x_{k+1} = g(x_k)$$

Equiv to $f(x) = 0$ form

$$g(x) = x \iff \underbrace{\boxed{g(x) - x}}_{f} = 0$$

$$f(x) = 0 \iff \underbrace{\boxed{f(x) + x}}_{g} = x$$