Lecture 33

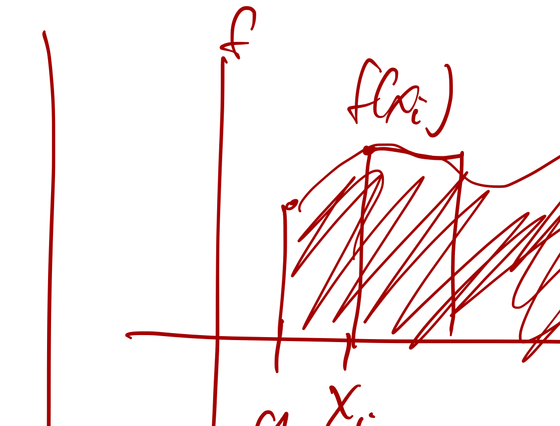
Integration!

$$\int_a^b f(x)\,dx \approx \boxed{???}$$

- Assume we have subroutine access to f.

How did we do this already?

Monte Carlo Approx



In MC, let X_i $(i=1,\dots,n)$ be uniform r.v. between a, b,

Then

$$\int_a^b f(x)dx \approx \frac{(b-a)}{n}\sum f(X_i)$$

We are not going to look @ those like you did in Calc!

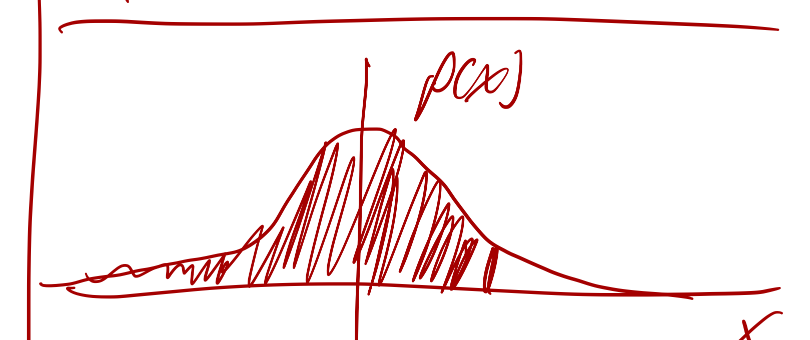
$$\int_a^b mx+c\,dx = \frac{mx^2}{2}+cx\Big|_a^b$$

$$\left(\frac{mb^2}{2}+cb\right)-\left(\frac{ma^2}{2}+ca\right)$$

Why do we need numerical integration?

- not always
- it's Easy (numerically) hard (analytically)
- if all we need is a #, why ~~spend~~ do algebra?
- Can't integrate analytically!

Example of no analytic formula



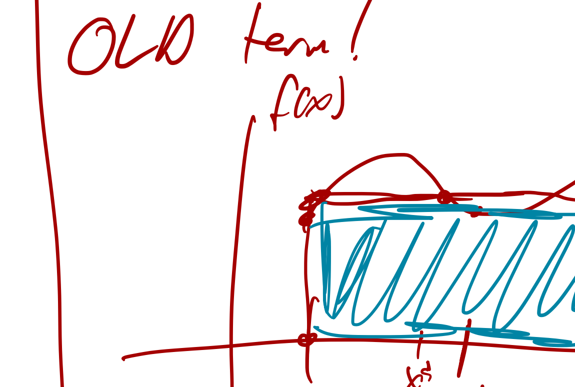
$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{2\pi} e^{-t^2/2}dt$$

$$= \frac{1}{2} + \boxed{\int_0^x \frac{1}{2\pi} e^{-t^2/2}dt}$$

Most use Num. Integration!

Numerical Integration is called $\boxed{\text{quadrature}}$

OLD term!



Idea:

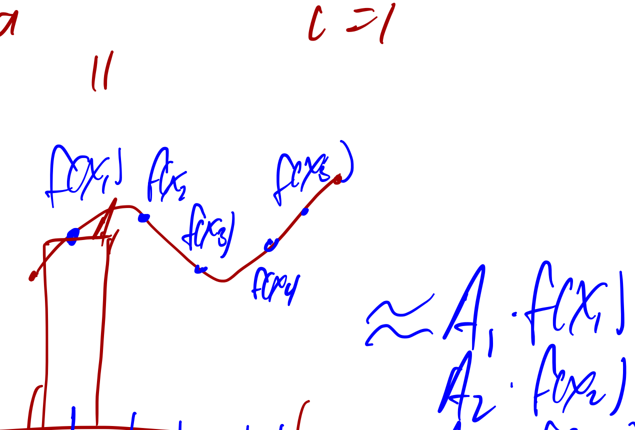
$$\int_a^b f(x)dx = A^* f(x^*)$$

Now a

Quadrature Rule is a num. approx. to an integral of the form

$$\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} A_i f(X_i)$$

$$\shortparallel$$



$$\approx A_1 \cdot f(X_1) + A_2 \cdot f(X_2) + A_3 \cdot f(X_3) + A_4 \cdot f(X_4) + A_5 \cdot f(X_5)$$

The values

X_i are called nodes

A_i are called weights

$X_i : i = 1$ to n

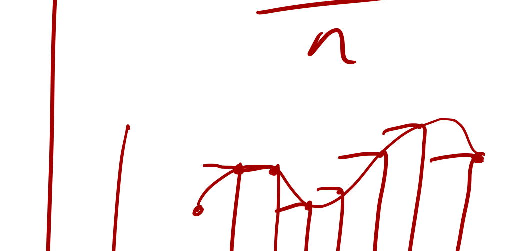
$A_i : i = 1$ to n

this gives 2n unknown "parameters" we can choose to accomplish various tools!

Example 1

$$X_i = \frac{(b-a)}{n}i + a$$

$$A_i = \frac{(b-a)}{n}$$



Example 2:

How to pick X_i, A_i to accomplish something!!

n = 1, goal: Ensure

$$\int_a^b f(x)dx = A_1 f(X_1)$$

is Exact for all linear functions!

Method of Undetermined Coefficients (§10.1)

$f(x) = mx + c$

$$\int_a^b mx+c\,dx = A_1 f(X_1)$$

$$= A_1(mX_1 + c)$$

When m = 0, it must work!

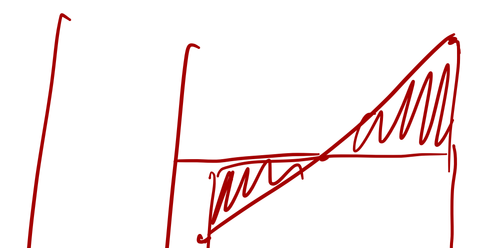
$$\int_a^b c\,dx = c(b-a) = \boxed{A_1 c}$$

$$\Rightarrow A_1 = b-a \quad 🙂$$

$$\int_a^b mx\,dx = \frac{mx^2}{2}\Big|_a^b = \frac{mb^2}{2} - \frac{ma^2}{2} = \frac{m}{2}(b-a)(b+a)$$

$$\boxed{= (b-a)mX_1}$$

$$X_1 = \frac{(b+a)}{2}$$



Other common classes of functions

- trig functions (sin, cos)
- exp
- polynomials!

For polys, there is an n-point formula that Exactly integrates polynomials of degree at most 2n-1

⇒ Called Gauss quadrature rules!

using ~~given~~ any set of distinct points between a, b, as the nodes

there is a quadrature rule that Exactly integrates polys of degree n-1

↳ if points are Chebyshev nodes ⇒ Called Clenshaw-Curtis quadrature → Approx Fun does this!

if pts are Unif. Space ⇒ Newton-Cotes quadrature.

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